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Some results of the driving point impedance functions

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Abstract **–** In this paper, a boundary version of the Schwarz lemma has been considered for driving point impedance functions at $s=0$ point of the imaginary axis. Accordingly, under $Z(0)=0$ condition, the modulus of the derivative of the $Z(s)$ function has been considered from below. Here, $Z(a)$ and $\frac{Z^{(p)}(a)}{p!}$ coefficients of the Taylor expansion of the $Z(s)=b+c_p(s-a)^p+\ldots$ function have been used in the obtained inequalities. The sharpness of these inequalities has also been proved. It is also shown that simple LC tank circuits and higher order filters are synthesized using the unique DPI functions obtained in each theorem.

Keywords – Schwarz lemma, Analytic function, Circuit.

I. INTRODUCTION

Driving point impedance (DPI) functions are positive real functions (PRF) which depend on the complex frequency parameter, s. A DPI function is physically realizable if it satisfies the properties of positive real functions which are given below [1]:

1-) $Z(s)$ is analytic and single valued in $\Re s\geq 0$ except possibly for poles on the axis of imaginaries,

2-) $Z(\bar{s})=\overline{Z(s)}$

3-) $\Re Z(s)\geq 0$, in $\Re s\geq 0$

In electrical engineering, derivative of positive real functions are mainly used in network analysis and synthesis [2, 3]. As one of the pioneer studies, Van Der Pol used the derivative of DPI functions to relate electric and magnetic energy [4]. Also, derivative of DPI functions are considered in [5]. In this study, Krueger and Brown established some new properties for DPI functions by specifically considering their derivatives. A gyrator based driving point impedance function is realized by Hazony in [6] by using again the derivative of DPI function and a new boundary is obtained by Reza for derivative of DPI functions noting they are also positive real functions [1]. Another boundary analysis made by Thomas Huang is presented in [7] for RL and RC DPI functions where the derivative is used to obtain the radius of the region mapped by the DPI function.

In this study, a boundary behavior of the module of the derivative of Driving point impedance (DPI) functions at the $s=0$ point on the imaginary axis is examined. That is, the derivative of the DPI function on the axis at the $s=0$ point on the imaginary axis, which is the boundary of the right-half plane, was investigated. Here, $Z(0)=0$ for Driving point impedance (DPI) function is accepted as $|Z'(0)|$ and the new inequalities have been obtained. $Z(s)$ functions from the nature of the problem have been found for the accuracy of the inequalities obtained for Driving point impedance (DPI) functions. To obtain these inequalities, the Schwarz lemma, which is one of the important issues of complex analysis, and the Schwarz lemma in the boundary of the unit disc, which is the result of it, are used.

II. PRELIMINARY CONSIDERATIONS

The most classical version of the Schwarz Lemma examines the behavior of a bounded, analytic function mapping the origin to the origin in the unit disc $D=\{\lambda:|\lambda|<1\}$. It is possible to see its effectiveness in the proofs of many important theorems. The Schwarz Lemma, which has broad applications and is the direct application of the maximum modulus principle, is given in the most basic form as follows ([10], p.329):

Let us consider a function $f(\lambda)=c_p\lambda^p+c_{p+1}\lambda^{p+1}+\ldots$ an analytic in the unit disc $D=\{\lambda:|\lambda|<1\}$ with $f(D)\subset D$. The Schwarz lemma asserts that

$$|f(\lambda)|\leq|\lambda|^p,$$

for every $\lambda \in D$ and

$$\left|c_p\right| \leq 1.$$

Moreover, if the equality $\left|f(\lambda)\right| = \left|\lambda\right|^p$ holds for any $\lambda \neq 0$, or $\left|c_p\right| = 1$ then f is a rotation, that is, $f(\lambda) = \lambda^p e^{i\theta}$, θ real.

Consider the function

$$f(\lambda) = \frac{Z(s)-b}{Z(s)+b}, \lambda = \frac{s-a}{s+a},$$

where $Z(s) = b + c_p(s-a)^p + c_{p+1}(s-a)^{p+1} + ..., p > 1$ and a, b are a positve real numbers. These mappings are also used in classical circuit theory [8, 1].

Here, $f(\lambda)$ is an analytic function in D, $f(0) = 0$ and $\left|f(\lambda)\right| < 1$ for $\left|\lambda\right| < 1$. Consider the auxiliary function

$$f(\lambda) = \frac{Z\left(a\frac{1+\lambda}{1-\lambda}\right)-b}{Z\left(a\frac{1+\lambda}{1-\lambda}\right)+b} = \frac{c_p\left(\frac{2a\lambda}{1-\lambda}\right)^p + c_{p+1}\left(\frac{2a\lambda}{1-\lambda}\right)^{p+1} + ...}{2b + c_p\left(\frac{2a\lambda}{1-\lambda}\right)^p + c_{p+1}\left(\frac{2a\lambda}{1-\lambda}\right)^{p+1} + ...}.$$

Then

$$\frac{f(\lambda)}{\lambda^p} = \frac{c_p\left(\frac{2a}{1-\lambda}\right)^p + c_{p+1}\left(\frac{2a}{1-\lambda}\right)^{p+1}\lambda + ...}{2b + c_p\left(\frac{2a\lambda}{1-\lambda}\right)^p + c_{p+1}\left(\frac{2a\lambda}{1-\lambda}\right)^{p+1} + ...}$$

and

$$d_p = \frac{(2a)^p}{2b} c_p.$$

Applying the Schwarz lemma for the function $f(\lambda)$, we obtain

$$\left|d_p\right| \leq 1$$

and

$$\left|c_p\right| \leq \frac{b}{a^p 2^{p-1}}.$$

This result is sharp with the function

$$Z(s) = b\frac{(s+a)^p + (s-a)^p}{(s+a)^p - (s-a)^p}.$$

We thus obtain the following lemma.

Lemma 1 *Let* $Z(s) = b + c_p(s-a)^p + c_{p+1}(s-a)^{p+1} + ..., p > 1$ *be a Positive Real Function. Then*

$$\left|c_p\right| \leq \frac{b}{a^p 2^{p-1}}. \tag{1.1}$$

The inequality (1.1) *is sharp, with equality for the function*

$$Z(s) = b\frac{(s+a)^p + (s-a)^p}{(s+a)^p - (s-a)^p}.$$

It is an elementary consequence of Schwarz lemma that if f extends continuously to some boundary point λ_0 with $\left|\lambda_0\right| = 1$, and if $\left|f(\lambda_0)\right| = 1$ and $f'(\lambda_0)$ exists, then $\left|f'(\lambda_0)\right| \geq 1$, which is known as the Schwarz lemma on the boundary. In [11], R. Osserman proposed the boundary refinement of the classical Schwarz lemma as follows:

Let $f : D \to D$ be an analytic function with $f(\lambda) = c_p\lambda^p + c_{p+1}\lambda^{p+1} + \dots$, $p \geq 1$. Assume that there is a $\lambda_0 \in \partial D$ so that f extends continuously to λ_0, $\left|f(\lambda_0)\right| = 1$ and $f'(\lambda_0)$ exists. Then

$$\left|f'(\lambda_0)\right| \geq p + \frac{1-\left|c_p\right|}{1+\left|c_p\right|}. \tag{1.2}$$

Thus, by the classical Schwarz lemma, it follows that

$$\left|f'(\lambda_0)\right| \geq p. \tag{1.3}$$

Inequality (1.2) is sharp. That is, for $\lambda_0 = 1$ in the inequality (1.3), equality occurs for the function $f(\lambda) = \lambda^p \frac{\lambda+\gamma}{1+\gamma\lambda}$, $\gamma \in [0,1]$. Also, $\left|f'(\lambda_0)\right| > p$ unless $f(\lambda) = \lambda^p e^{i\theta}$, θ real. Inequality (1.6) and its generalizations have important applications in geometric theory of functions and they are still hot topics in the mathematics literature [9, 12, 13].

III. Main Results

In this section, a boundary version of the Schwarz lemma has been considered for driving point impedance functions at $s = 0$ point of the imaginary axis. Accordingly, under $Z(0) = 0$ condition, the modulus of the derivative of the $Z(s)$ function has been considered from below. Here, $Z(a)$ and $\frac{Z^{(p)}(a)}{p!}$ coefficients of the Taylor expansion of the

$Z(s) = b + c_p(s - a)^p + \ldots$ function have been used in the obtained inequalities. The sharpness of these inequalities has also been proved.

Theorem 1 *Let* $Z(s) = b + c_p(s-a)^p + c_{p+1}(s-a)^{p+1} + \ldots,\ p > 1$ *be a positive real function that is also analytic at the point* $s = 0$ *of the imaginary axis with* $Z(0) = 0$. *Then*

$$\left|Z'(0)\right| \geq \frac{b}{a} p. \tag{2.1}$$

The results (2.1) *is sharp for the function given by*

$$Z(s) = \begin{cases} b\dfrac{(s+a)^p + (s-a)^p}{(s+a)^p - (s-a)^p}, & p = 3,5,\ldots,n \\ b\dfrac{(s+a)^p - (s-a)^p}{(s+a)^p + (s-a)^p}, & p = 2,4,6,\ldots,n \end{cases}.$$

Proof. Let

$$f(\lambda) = \frac{Z\left(a\dfrac{1+\lambda}{1-\lambda}\right) - b}{Z\left(a\dfrac{1+\lambda}{1-\lambda}\right) + b}.$$

with the simple calculations, we have

$$f'(\lambda) = \frac{\dfrac{4ab}{(1-\lambda)^2} Z'\left(a\dfrac{1+\lambda}{1-\lambda}\right)}{\left(Z\left(a\dfrac{1+\lambda}{1-\lambda}\right) + b\right)^2}.$$

Therefore, from (1.3), we obtain

$$p \leq \left|f'(-1)\right| = \frac{ab\left|Z'(0)\right|}{\left|Z(0)+b\right|^2} = \frac{a}{b}\left|Z'(0)\right|$$

and

$$\left|Z'(0)\right| \geq \frac{b}{a} p.$$

Now, we shall show that the inequality (2.1) is sharp. Let

$$Z(s) = b\frac{(s+a)^p + (s-a)^p}{(s+a)^p - (s-a)^p}.$$

Then

$$Z'(s) = b\frac{\left(p(s+a)^{p-1} + p(s-a)^{p-1}\right)\left((s+a)^p - (s-a)^p\right)}{\left((s+a)^p - (s-a)^p\right)^2}$$

$$-b\frac{\left(p(s+a)^{p-1} - p(s-a)^{p-1}\right)\left((s+a)^p + (s-a)^p\right)}{\left((s+a)^p - (s-a)^p\right)^2},$$

$$Z'(0) = b\frac{\left(pa^{p-1} + p(-a)^{p-1}\right)\left(a^p - (-a)^p\right)}{\left(a^p - (-a)^p\right)^2}$$

$$-b\frac{\left(pa^{p-1} - p(-a)^{p-1}\right)\left(a^p + (-a)^p\right)}{\left(a^p - (-a)^p\right)^2},$$

and

$$\left|Z'(0)\right| = b\frac{4p(a)^{2p-1}}{\left|a^p - (-a)^p\right|^2}.$$

Therefore, since $p = 3,5,\ldots,n$, (2.1) is satisfied with equality. Similarly, for $Z(s) = b\dfrac{(s+a)^p - (s-a)^p}{(s+a)^p + (s-a)^p}$, $p = 2,4,6,\ldots,n$, (2.1) is satisfied with equality.

For DPI impedances Cauer 1 realization is used. Let us consider the case where *p = 3* in Theorem 1. For *p = 3* the Z(s) function is given

$$Z(s) = b\frac{s^3 + 3a^2 s}{3as^2 + a^3}$$

And corresponding circuit is as shown in Fig.1.

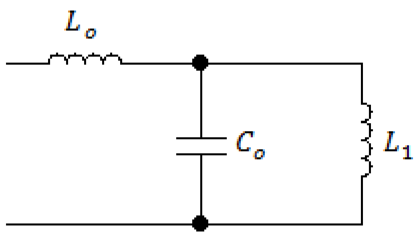

Figure 1. Corresponding circuit for $p = 3$ in Theorem 1 where $L_o = \frac{3b}{a}H,\ C_o = \frac{8}{9ab}F,\ L_1 = \frac{3b}{8a}H.$

The generalized circuit model of the impedance function obtained for a odd value of $p = n$ is given

$$Z(s) = b\frac{b_n s^n + b_{n-2}s^{n-2} + \cdots + b_1 s}{s^{n-1} + a_{n-3}s^{n-3} + \cdots + a_0}$$

and corresponding circuit is as shown in Fig.2

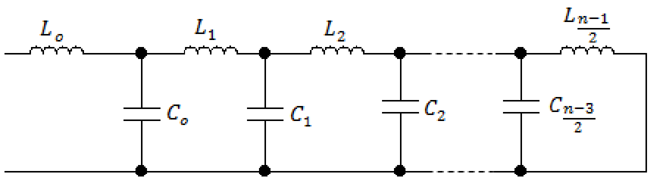

Figure 2. The generalized circuit model of the impedance function obtained for odd $p = n$ in Theorem 1.

Let us consider the case where $p = 2$ in Theorem 1. For $p = 2$ the Z(s) function is given

$$Z(s) = b\frac{2as}{s^2 + a^2}$$

and corresponding circuit is as shown in Fig.3.

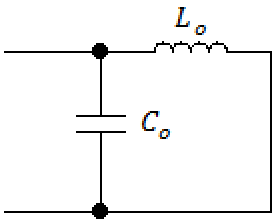

Figure 3. Corresponding circuit for $p = 2$ in Theorem 1 where $L_o = \frac{2b}{a}H,\ C_o = \frac{1}{2ab}F.$

The generalized circuit model of the impedance function obtained for a even value of $p = n$ is given

$$Z(s) = b\frac{b_{n-1}s^{n-1} + b_{n-3}s^{n-3} + \cdots + b_1 s}{a_n s^n + a_{n-2}s^{n-2} + \cdots + a_0}$$

and corresponding circuit is as shown in Fig.4.

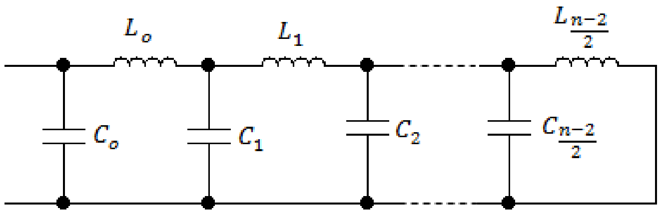

Figure 4. The generalized circuit model of the impedance function obtained for even $p = n$ in Theorem 1.

Theorem 2 *Let* $Z(s) = b + c_p(s-a)^p + c_{p+1}(s-a)^{p+1} + \ldots,\ p > 1$ *be a positive real function that is also analytic at the point* $s = 0$ *of the imaginary axis with* $Z(0) = 0$. *Then*

$$\left|Z^{'}(0)\right| \geq \frac{b}{a}\left(p + \frac{2b - (2a)^p\left|c_p\right|}{2b + (2a)^p\left|c_p\right|}\right). \qquad (2.2)$$

The results (2.2) *is sharp for the function given by*

$$Z(s) = \begin{cases} b\dfrac{(s+a)^{p+1} - c(s-a)(s+a)^p - (s-a)^{p+1} + c(s-a)^p(s+a)}{(s+a)^{p+1} - c(s-a)(s+a)^p + (s-a)^{p+1} - c(s-a)^p(s+a)}, & p = 3,5,\ldots,n \\ b\dfrac{(s+a)^{p+1} - c(s-a)(s+a)^p + (s-a)^{p+1} - c(s-a)^p(s+a)}{(s+a)^{p+1} - c(s-a)(s+a)^p - (s-a)^{p+1} + c(s-a)^p(s+a)}, & p = 2,4,6,\ldots,n \end{cases},$$

where $c = \dfrac{2^{p-1}a^p}{b}\left|c_p\right|$ *is an arbitrary number from* $[0,1]$ (*see*, (1.1)).

Proof. Let $f(\lambda)$ be the same as in the proof of Theorem 1. Therefore, from (1.2), we take

$$p + \frac{1-\left|a_p\right|}{1+\left|a_p\right|} \leq \left|f^{'}(-1)\right| = \frac{ab\left|Z^{'}(0)\right|}{\left|Z(0)+b\right|^2} = \frac{a}{b}\left|Z^{'}(0)\right|,$$

where $\left|a_p\right| = \left|\dfrac{f^{(p)}(\alpha)}{p!}\right|$.

Since

$$f(\lambda) = \frac{Z\left(\alpha\dfrac{1+\lambda}{1-\lambda}\right) - b}{Z\left(\alpha\dfrac{1+\lambda}{1-\lambda}\right) + b} = \frac{c_p\left(\dfrac{2a\lambda}{1-\lambda}\right)^p + c_{p+1}\left(\dfrac{2a\lambda}{1-\lambda}\right)^{p+1} + \ldots}{2b + c_p\left(\dfrac{2az}{1-\lambda}\right)^p + c_{p+1}\left(\dfrac{2a\lambda}{1-\lambda}\right)^{p+1} + \ldots},$$

$$\frac{f(\lambda)}{\lambda^p} = \frac{c_p\left(\dfrac{2a}{1-\lambda}\right)^p + c_{p+1}\left(\dfrac{2a}{1-\lambda}\right)^{p+1}\lambda + \ldots}{2b + c_p\left(\dfrac{2a\lambda}{1-\lambda}\right)^p + c_{p+1}\left(\dfrac{2a\lambda}{1-\lambda}\right)^{p+1} + \ldots}$$

and

$$\left|a_p\right| = \frac{2^{p-1}a^p}{b}\left|c_p\right|,$$

we obtain

$$p + \frac{1 - \dfrac{2^{p-1}a^p}{b}\left|c_p\right|}{1 + \dfrac{2^{p-1}a^p}{b}\left|c_p\right|} \leq \frac{a}{b}\left|Z^{'}(0)\right|$$

and

$$\left|Z^{'}(0)\right| \geq \frac{b}{a}\left(p + \frac{2b - 2^p a^p\left|c_p\right|}{2b + 2^p a^p\left|c_p\right|}\right).$$

Now, we shall show that the inequality (2.2) is sharp. Let

$$Z(s) = b\frac{(s+a)^{p+1} - c(s-a)(s+a)^p - (s-a)^{p+1} + c(s-a)^p(s+a)}{(s+a)^{p+1} - c(s-a)(s+a)^p + (s-a)^{p+1} - c(s-a)^p(s+a)}.$$

From the equation given above, we have

$$Z^{'}(s) = -4ab\frac{(s+a)^p}{a^2 - s^2}\frac{(s-a)^p\left(a^2 - c^2a^2 - ps^2 + pa^2 + c^2s^2 - s^2 + c^2p\alpha^2 + 2cps^2 + 2cp\alpha^2 - c^2ps^2\right)}{\left((s+a)^{p+1} + (s-a)^{p+1} - c(s+a)(s-a)^p - c(s+a)^p(s-a)\right)^2}$$

and

$$|Z'(0)| = 4ab\frac{(a)^p}{a^2}\frac{(-a)^p\left(a^2 - c^2a^2 + pa^2 + c^2pa^2 + 2cpa^2\right)}{\left((a)^{p+1} + (-a)^{p+1} - ca(-a)^p - c(a)^p(-a)\right)^2}.$$

Since $c = \frac{2^{p-1}a^p}{b}|c_p|$, for $p = 3,5,\ldots,n$, (2.2) is satisfied with equality.

Similarly, for $p = 2,4,6,\ldots,n$,

$$Z(s) = b\frac{(s+a)^{p+1} - c(s-a)(s+a)^p + (s-a)^{p+1} - c(s-a)^p(s+a)}{(s+a)^{p+1} - c(s-a)(s+a)^p - (s-a)^{p+1} + c(s-a)^p(s+a)},$$

(2.2) is satisfied with equality.

Let us consider the case where *p = 3* in Theorem 2. For *p = 3* the Z(s) function is given

$$Z(s) = b\frac{(8a-4ac)s^3 + (8a^3 + 4a^3c)s}{(2-2c)s^4 + 12a^2s^2 + (2a^4 + 2ca^4)} = b\frac{b_3s^3 + b_1s}{a_4s^4 + a_2s^2 + a_0}$$

and corresponding circuit is as shown in Fig.5.

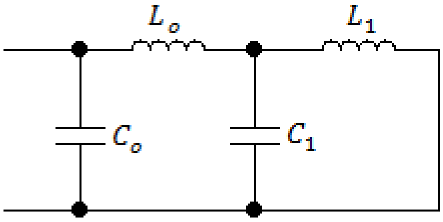

Figure 5. Corresponding circuit for *p = 3* in Theorem 2.

The generalized circuit model of the impedance function obtained for a odd value of $p = n$ is given

$$Z(s) = b\frac{b_ns^n + b_{n-2}s^{n-2} + \cdots + b_1s}{a_{n+1}s^{n+1} + a_{n-1}s^{n-1} + \cdots + a_0}$$

And corresponding circuit is as shown in Fig.6.

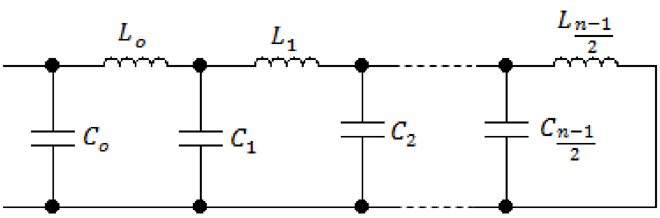

Figure 6. The generalized circuit model of the impedance function obtained for odd *p = n* in Theorem 2.

Let us consider the case where *p = 2* in Theorem 2. For *p = 2* the Z(s) function is given

$$Z(s) = b\frac{(2-2ac)s^3 + (6a^2 + 2a^2c)s}{(6a-2ac)s^2 + (2a^3 + 2a^3c)} = b\frac{b_3s^3 + b_1s}{a_2s^2 + a_0}$$

and corresponding circuit is as shown in Fig.7.

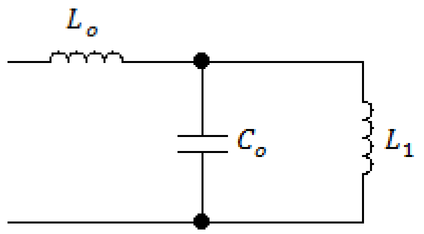

Figure 7. Corresponding circuit for *p = 2* in Theorem 2.

The generalized circuit model of the impedance function obtained for a even value of $p = n$ is given

$$Z(s) = b\frac{b_{n+1}s^{n+1} + b_{n-1}s^{n-1} + \cdots + b_1s}{a_ns^n + a_{n-2}s^{n-2} + \cdots + a_0}$$

And corresponding circuit is as shown in Fig.8.

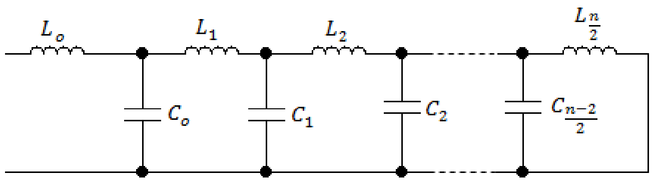

Figure 8. The generalized circuit model of the impedance function obtained for even p = *n* in Theorem 2.

IV. CONCLUSION

In this study, boundary version of the Schwarz lemma has been analysed to obtain novel driving point impedance functions. Under Z(0)=0 condition, the modulus of the derivative of Z(s) at s=0, that is |Z'(0)| has been considered from below. In Theorems 1 and 2, it has been shown that lower boundaries can be obtained for |Z'(0)|. The extremal functions corresponding to these inequalities have also been presented. These extremal functions are interpreted as driving point impedance functions in aspect of electrical engineering and dependence of these functions on the p parameter makes it possible to design circuits with different number of inductors and capacitors. For each theorem, general circuit schematics for odd and even values of p parameter have been obtained, respectively. Considering these circuits, it is possible to say that different filter structures can be proposed using the presented theorem within this study.

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