ISAS 2018-Winter

2^{nd}International Symposium on Innovative Approaches in Scientific Studies

SETSCI Conference Indexing System,
Volume 3 (2018), 313-316

November 30 – December 2, 2018, Samsun, Turkey

GENERALIZATIONS OF FIXED POINT THEOREMS FOR MULTIVALUED MAPS VIA Q-FUNCTIONS

NURCAN BILGILI GUNGOR

ABSTRACT. In 1996 Kada, Suzuki and Takahashi defined the w-distance mappings on metric spaces and they proved fixed point theorems for w-distances. In 2008 Homidan, Ansari and Yao gave the Q-functions on quasi-metric space and they generalized the main results of Kada et al., since every w-distance is a Q-function. In 2011 Marin, Romaguera and Tirado introduced the generalization of Q-functions to T_0 quasipseudometric spaces and they gave a new fixed point theorem for T_0 quasipseudometric spaces by using Bianchini-Grandolfi gauge functions. In this paper the generalization of fixed point theorems for multivalued maps via Q-functions on complete T_0 quasipseudometric spaces are investigated. Also, the conclusions related to previous theorems in this field are given.

1. Introduction and Preliminaries

In this paper the set of positive integer numbers and the set of nonnegative integer numbers will be denoted the latter $\mathbb{N}^+$ and N, respectively.

By a T_0 quasipseudometric on a set X, we mean a function $d : X \times X \to [0, \infty)$ such that for all $a, b, c \in X$,
(T_0qpm1) $d(a, b) = d(b, a) = 0 \Leftrightarrow a = b$,
(T_0qpm2) $d(a, c) \leq d(a, b) + d(b, c)$.

A T_0 quasipseudometric d on X that satisfies the condition
$(T_0qpm1^{'})$ $d(a, b) = 0 \Leftrightarrow a = b$
instead of $(T_0qpm1)d(a, b) = d(b, a) = 0 \Leftrightarrow a = b$,, then it is called a quasimetric on X.

In the sequel we will use T_0 qpm instead of T_0 quasipseudometric. If d is a T_0 qpm on X, then (X, d) is called quasipseudometric space and if d is a quasimetric on X, then (X, d) is called quasimetric space.

Given a T_0 qpm d on a set X, the function d^{-1} defined by $d^{-1}(a, b) = d(b, a)$, is also a T_0 qpm on X, called the conjugate of d, and the function d^s defined by $d^s(x, y) = \max\{d(a, b), d^{-1}(b, a)\}$ is a metric on X, called the supremum metric associated to d.

Thus, every T_0 qpm d on X induces, in a natural way, three topologies denoted by $\tau_d, \tau_{d_{-1}}$ and τ_{d^s}, respectively, and defined as follows.
(τ_i) τ_d is the T_0 topology on X which has a base the family of τ_d- open balls $\{B_d(a, \varepsilon) : a \in X, \varepsilon > 0\}$, where $B_d(a, \varepsilon) = \{b \in X : d(a, b) < \varepsilon\}$, for all $a \in X$ and $\varepsilon > 0$.
(τ_{ii}) $\tau_{d^{-1}}$ is the T_0 topology on X which has a base the family of $\tau_{d^{-1}}-$ open balls $\{B_{d^{-1}}(a, \varepsilon) : a \in X, \varepsilon > 0\}$, where $B_{d^{-1}}(a, \varepsilon) = \{b \in X : d_{-1}(a, b) < \varepsilon\}$, for all $a \in X$ and $\varepsilon > 0$.
(τ_{iii}) τ_{d^s} is the topology on X induced by the metric d^s.

2000 *Mathematics Subject Classification.* 47H10, 54H25.

Key words and phrases. fixed point, w-distance, Q-function, quasi-metric space , quasipseudometric space, T_0 quasipseudometric space, multivalued maps.

Note that if d is quasimetric on X, then d^{-1} is also a quasimetric, and τ_d and $\tau_{d^{-1}}$ are T_1 topologies on X.

Note also that a sequence $(a_n)_{n\in\mathbb{N}}$ in a T_0 qpm space (X,d) is $\tau_d - convergent$ (respectively, $\tau_{d^{-1}} - convergent$) to $a \in X$ if and only if $\lim_{n\to\infty} d(a,a_n) = 0$ (respectively, $\lim_{n\to\infty} d(a_n,a) = 0$).

A T_0 qpm space (X,d) is said to be complete if every Cauvhy sequence is $\tau_{d^{-1}} - convergent$, where a sequence $(a_n)_{n\in\mathbb{N}}$ is called Cauchy if for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}^+$ such that $d(a_n,a_m) < \varepsilon$ whenever $m \geq n \geq n_\varepsilon$.

In this case, we say that d is a complete T_0 qpm on X.

Definition 1.1. *A $Q-$ function on a T_0 qpm space (X,d) is a function $Q : X \times X \to [0,\infty)$ satisfying the following conditions:*
$(Q1) Q(a,c) \leq Q(a,b) + Q(b,c)$, for all $a,b,c \in X$,
$(Q2)$ if $a \in X, M > 0$, and $(b_n)_{n\in\mathbb{N}}$ is a sequence in X that $\tau_{d^{-1}} -$ converges to a point $b \in X$ and satisfies $Q(a,b_n) \leq M$, for all $n \in \mathbb{N}$, then $Q(a,b) \leq M$,
$(Q3)$ for each $\varepsilon > 0$ there exists $\delta > 0$ such that $Q(a,b) \leq \delta$ and $Q(a,c) \leq \delta$ imply $d(b,c) \leq \varepsilon$.

Lemma 1.1. *Let Q be a $Q - function$ on a T_0 qpm space (X,d). Then, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $Q(a,b) \leq \delta$ and $Q(a,c) \leq \delta$ imply $d^s(b,c) \leq \varepsilon$.*

Definition 1.2. *A partial metric on a set X is a function $p : X \times X \to [0,\infty)$ such that, for all $a,b,c \in X$:*
$(P1) a = b \Leftrightarrow p(a,a) = p(a,b) = p(b,b)$,
$(P2) p(a,a) \leq p(a,b)$,
$(P3) p(a,b) = p(b,a)$,
$(P4) p(a,c) \leq p(a,b) + p(b,c) - p(b,b)$.

Given a T_0 qpm space (X,d), we denote by 2^X the collection of all nonempty subsets of X, by $Cl_{d^{-1}}(X)$ the collection of all nonempty $\tau_{d^{-1}} - closed$ subsets of X, and by $Cl_{d^s}(X)$ the collection of all nonempty $\tau_{d^s} - closed$ subsets of X.

2. Main Results

Now, we are ready to give and prove our main results.

Theorem 2.1. *Let (X,d) be a complete T_0 qpm space, Q a $Q-$ function on X, and $T : X \to Cl_{d^s}(X)$ a multivalued map such that for each $a,b \in X$ and $u \in T(a)$, there is $v \in T(b)$ satisfying*

$$Q(u,v) \leq \phi(\max\{Q(a,b), Q(a,u), Q(b,v)\}), \tag{1}$$

where $\phi : [0,\infty) \to [0,\infty)$ is a Bianchini-Grandolfi gauge function.Then, there exists $c \in X$ such that $c \in T(c)$ and $Q(c,c) = 0$.

Proof. Fix $a_0 \in X$ and $a_1 \in T(x_0)$. If $a_1 = a_0$ then we have nothing to prove. By hypothesis, there exists $a_2 \in T(a_1)$ such that

$$Q(a_1,a_2) \leq \phi(\max\{Q(a_0,a_1), Q(a_1,a_2)\}). \tag{2}$$

If $a_2 = a_1$ then proof is completed. So we assume that $a_2 \neq a_1$.

If $\max\{Q(a_0,a_1),Q(a_1,a_2)\}=Q(a_1,a_2)$ then $Q(a_1,a_2)\leq\phi(Q(a_1,a_2)<Q(a_1,a_2)$. This is a contradiction. So $\max\{Q(a_0,a_1),Q(a_1,a_2)\}=Q(a_0,a_1)$ and $Q(a_1,a_2)\leq\phi(Q(a_0,a_1)$. Similarly we continuing this process, there exists $a_3\in T(a_2)$ such that

$$Q(a_2,a_3)\leq\phi(\max\{Q(a_1,a_2),Q(a_2,a_3)\}). \tag{3}$$

If $a_3=a_2$ then proof is completed. So we assume that $a_3\neq a_2$.

If $\max\{Q(a_1,x_2),Q(a_3,a_2)\}=Q(a_2,a_3)$ then $Q(a_2,a_3)\leq\phi(Q(a_2,a_3)<Q(a_2,a_3)$. This is a contradiction. So $\max\{Q(a_1,a_2),Q(a_2,a_3)\}=Q(a_1,a_2)$ and $Q(a_2,a_3)\leq\phi(Q(a_1,a_2)$. From ϕ is a Bianchini-Grandolfi gauge function $Q(a_2,a_3)\leq\phi(Q(a_1,a_2)\leq\phi^2(Q(a_0,a_1))$. Following this process, we get a sequence $(a_n)_{n\in w}$ with $a_n\in T(a_{n-1})$ and $Q(a_n,a_{n+1})\leq\phi(Q(a_{n-1},a_n))$, for all $n\in\mathbb{N}^+$. Therefore

$$Q(a_n,a_{n+1})\leq\phi^n(Q(a_0,a_1)), \tag{4}$$

for all $n\in\mathbb{N}^+$. Then, choose $\varepsilon>0$. Let $\delta=\delta(\varepsilon)\in(0,\varepsilon)$ is satisfy condition (Q_3). We will get that there is $n_\delta\in\mathbb{N}$ such that $Q(a_n,a_m)<\delta$ whenever $m>n\geq n_\delta$. Indeed, if $Q(a_0,a_1)=0$, then $\phi(Q(a_0,a_1)=0$ and thus $Q(a_n,a_{n+1})=0$, for all $n\in\mathbb{N}^+$, then by condition (Q_1), $Q(a_n,a_m)=0$ whenever $m>n$. If $Q(a_0,a_1)>0$, $\sum_{n=0}^{\infty}\phi^n(Q(a_0,a_1))<\infty$, so there is $n_\delta\in\mathbb{N}^+$ such that

$$\sum_{n=n_\delta}^{\infty}\phi^n(Q(a_0,a_1))<\delta. \tag{5}$$

Then, for $m>n>n_\delta$, we obtain

$$\begin{aligned} Q(a_n,a_m) &\leq Q(a_n,a_{n+1})+Q(a_{n+1},a_{n+2})+...+Q(a_{m_1},a_m)\\ &\leq\phi^n(Q(a_0,a_1))+\phi^{n+1}(Q(a_0,a_1))+...+\phi^{m-1}(Q(a_0,a_1))\\ &\leq\textstyle\sum_{j=n_\delta}^{\infty}\phi^j(Q(a_0,a_1))<\delta. \end{aligned} \tag{6}$$

In particular, $Q(a_{n_\delta},a_n)\leq\delta$ and $Q(a_{n_\delta},a_m)\leq\delta$ whenever $n,m>n_\delta$, thus, by Lemma 1.1 , $d^s(a_n,a_m)\leq\varepsilon$ whenever $n,m>n_\delta$. So $(a_n)_{n\in w}$ is a Cauchy sequence in (X,d^s), and then it is simple to see $(a_n)_{n\in w}$ is a Cauchy sequence in (X,d). Since (X,d) is complete there exists $c\in X$ such that $\lim_{n\to\infty}d(a_n,c)=0$. Next, we obtain that $c\in T(c)$. First we prove that $\lim_{n\to\infty}Q(a_n,c)=0$. Indeed, $\varepsilon>0$ is given. Fix $n\geq n_\delta$. Since $Q(a_n,a_m)\leq\delta$ whenever $m>n$, we get from condition (Q_2) that $Q(a_n,c)\leq\delta<\varepsilon$ whenever $n\geq n_\delta$. Then we choose for each $n\in\mathbb{N}^+$ take $b_n\in T(c)$ such that

$$Q(a_n,b_n)\leq\phi(\max\{Q(a_{n-1},c),Q(a_{n-1},a_n),Q(c,b_n)\}). \tag{7}$$

In the sequel, the way similar to the previous part of the proof is used, $\lim_n Q(a_n,b_n)=0$ is obtained and by Lemma 1.1,

$$\lim_{n\to\infty}d^s(c,b_n)=0. \tag{8}$$

So, $c\in Cl_{d^s}(T(c))=T(c)$. It remains to prove that $Q(c,c)=0$. Because of $c\in T(c)$, we can get a sequence $(c_n)_{n\in\mathbb{N}}$ in X such that $c_1\in T(c),c_{n+1}\in T(c_n)$ and

$$Q(c,c_n)\leq\phi(\max\{Q(c,c_{n-1}),Q(c,c),Q(c_{n-1},c_n)\}),\forall n\in\mathbb{N}^+. \tag{9}$$

Then, for the all cases we obtain $\lim_{n\to\infty}Q(c,c_n)=0$. So, by using Lemma 1.1, $(c_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d)(in fact, it is a Cauchy sequence in (X,d^s)). Let $u\in X$ such that $\lim_{n\to\infty}d(c_n,u)=0$. Given $\varepsilon>0$, there is $n_\varepsilon\in\mathbb{N}$ such that $Q(c,c_n)<\varepsilon$, for all $n\geq n_\varepsilon$. By using condition (Q_2), we deduce that $Q(c,u)\leq\varepsilon$, then $Q(c,u)=0$. Since $\lim_{n\to\infty}Q(a_n,c)=0$, it follows from condition (Q_1) that $\lim_{n\to\infty}Q(a_n,u)=0$. Therefore, $d^s(c,u)\leq\varepsilon\leq\varepsilon$, for all $\varepsilon>0$, from condition (Q_3). And we get that $c=u$, thus $Q(c,c)=0$. □

Corollary 2.1. *(Theorem 3.3 in [4]) Let (X,d) be a complete T_0 qpm space, Q a $Q-$ function on X, and $T:X\to Cl_{d^s}(X)$ a multivalued map such that for each $a,b\in X$ and $u\in T(a)$, there is $v\in T(b)$ satisfying*

$$Q(u,v) \leq \phi(Q(a,b)), \tag{10}$$

where $\phi : [0,\infty) \to [0,\infty)$ *is a Bianchini-Grandolfi gauge function.Then, there exists* $c \in X$ *such that* $c \in T(c)$ *and* $Q(c,c) = 0$.

Corollary 2.2. *(Corollary 3.5 in [4]) Let* (X,p) *be a partial metric space such that the induced weightable* T_0 *qpm space* d_p *is complete and* $T : X \to Cl_{d^s}(X)$ *a multivalued map such that for each* $a,b \in X$ *and* $u \in T(a)$*, there is* $v \in T(b)$ *satisfying*

$$p(u,v) \leq \phi(p(a,b)), \tag{11}$$

where $\phi : [0,\infty) \to [0,\infty)$ *is a Bianchini-Grandolfi gauge function. Then, there exists* $c \in X$ *such that* $c \in T(c)$ *and* $Q(c,c) = 0$.

References

[1] Al-Homidan, S., Ansari, Q. H. and Yao, J. C., Some generalizations of Ekeland-type variational principle with applications to equilibrium problems and fixed point theory. Nonlinear Analysis: Theory, Methods and Applications, 69(1), 2008, 126-139.

[2] Kada, O., Suzuki, T. and Takahashi, W., Nonconvex minimization theorems and fixed point theorems in complete metric spaces. Mathematica japonicae, 44(2), 1996, 381-391.

[3] Latif, A. and Al-Mezel, S. A., Fixed point results in quasimetric spaces. Fixed Point Theory and Applications, 2011(1), 2011, 178306.

[4] Marn, J., Romaguera, S. and Tirado, P., Q-Functions on Quasimetric Spaces and Fixed Points for Multivalued Maps. Fixed Point Theory and Applications, 2011(1), 2011, 603861.

Nurcan Bilgili Gungor,
Amasya University,
Department of Mathematics, Faculty of Science and Arts, 05100, Amasya, Turkey
E-mail address: bilgilinurcan@gmail.com, nurcan.bilgili@amasya.edu.tr