SETSCI Conference Proceedings
4 (6), 16-18, 2019

4th International Symposium on Innovative Approaches in Engineering and Natural Sciences

November 22-24, 2019, Samsun, Turkey

https://doi.org/10.36287/setsci.4.6.011
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ISAS WINTER - 2019

ON SOME PROPERTIES OF A VECTOR-VALUED FUNCTION SPACE

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Presentation/Paper Type: Oral / Full Paper

Abstract **–** The study of various mathematical problems (such as elasticity, non-Newtonian fluids and electrorheological fluids) with variable exponent growth condition has been received considerable attention in recent years. These problems are interesting in applications and raise many difficult mathematical problems. There are also a lot of published papers in these spaces. Vector-valued Lebesgue spaces are widely used in analysis, abstract evolution equations and in the theory of integral operators. In this paper we recall the weighted vector-valued classical and variable exponent Lebesgue spaces. We define a intersection space of vector-valued weighted classical Lebesgue and variable exponent Lebesgue spaces. We discuss some basic properties, such as, Banach space, dense subspaces and Hölder type inequalities of these spaces. We will also show that every elements of vector-valued these spaces are locally integrable. Moreover, we investigate several embeddings and continuous embeddings properties of these spaces under some conditions with respect to exponents and two weight functions.

Keywords *– Vector-valued weighted classical and variable exponent Lebesgue spaces, locally integrable, continuous embedding*

I. INTRODUCTION

The variable exponent Lebesgue space $L^{p(.)}(\mathbb{R}^n)$ and Sobolev space $W^{k,p(.)}(\mathbb{R}^n)$ were introduced by Kováčik and Rákosník [10] In 1991. Since 1991, variable exponent Lebesgue, Sobolev, Besov, Triebel-Lizorkin, Lorentz, amalgam and Morrey spaces, have attracted many attentions (see [10], [5], [7], [6]). Vector-valued variable exponent Bochner-Lebesgue spaces $L^{p(.)}(\mathbb{R}^n, E)$ defined by Cheng and Xu [4] in 2013. They proved dual space, the reflexivity, uniformly convexity and uniformly smoothness of $L^{p(.)}(\mathbb{R}^n, E)$. Furthermore, they gave some properties of the Banach valued Bochner-Sobolev spaces with variable exponent.

II. PRELIMINARIES

Definition 1. For a measurable function $p:\mathbb{R}^n \to [1,\infty)$ (called a variable exponent on $\mathbb{R}^n$), we put

$$p^- = essinf_{x\in\mathbb{R}^n} p(x), \qquad p^+ = esssup_{x\in\mathbb{R}^n} p(x).$$

The variable exponent Lebesgue spaces $L^{p(.)}(\mathbb{R}^n)$ consist of all measurable functions f such that $\rho_{p(.)}(\gamma f) < \infty$ for some $\gamma > 0$, equipped with the Luxemburg norm

$$\|f\|_{p(.)} = inf\left\{\gamma > 0 : \rho_{p(.)}\left(\frac{f}{\gamma}\right) \le 1\right\},$$

where

$$\rho_{p(.)}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx.$$

If $p^+ < \infty$, then $f \in L^{p(.)}(\mathbb{R}^n)$ iff $\rho_{p(.)}(f) < \infty$. The space $\left(L^{p(.)}(\mathbb{R}^n), \|.\|_{p(.)}\right)$ is a Banach space. If $p(.) = p$ is a constant function, then the norm $\|.\|_{p(.)}$ coincides with the usual Lebesgue norm $\|.\|_p$ [10]. In this paper we assume that $p^+ < \infty$.

A positive, measurable and locally integrable function $\vartheta:\mathbb{R}^n \to (0,\infty)$ is called a weight function. The weighted modular is defined by

$$\rho_{p(.),\vartheta}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} \vartheta(x) dx.$$

The weighted variable exponent Lebesgue space $L^{p(.)}_{\vartheta}(\mathbb{R}^n)$ consists of all measurable functions f on $\mathbb{R}^n$ for which $\|f\|_{p(.),\vartheta} = \left\|f\vartheta^{\frac{1}{p(.)}}\right\|_{p(.)} < \infty$.

Theorem 2. Let $\frac{1}{p(.)} + \frac{1}{q(.)} = 1$ and $\vartheta^* = \vartheta^{1-q(.)}$. Then for $f \in L^{p(.)}_{\vartheta}(\mathbb{R}^n)$ and $g \in L^{p(.)}_{\vartheta^*}(\mathbb{R}^n)$, we have $fg \in L^1(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \le C\|f\|_{p(.),\vartheta} \|g\|_{p(.),\vartheta^*}$$

for some $C > 0$.

Proof. By the Hölder inequality for variable exponent Lebesgue spaces, we get

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx = \int_{\mathbb{R}^n} |f(x)g(x)| \vartheta^{\frac{1}{p(.)} - \frac{1}{p(.)}} dx$$
$$\le C \left\|f\vartheta^{\frac{1}{p(.)}}\right\|_{p(.)} \left\|g\vartheta^{-\frac{1}{p(.)}}\right\|_{q(.)}$$

for some $C > 0$. That is the desired result.

So the dual space of $L^{p(.)}_{\vartheta}(\mathbb{R}^n)$ is $L^{p(.)}_{\vartheta^*}(\mathbb{R}^n)$, where $\frac{1}{p(.)} + \frac{1}{q(.)} = 1$ and $\vartheta^* = \vartheta^{1-q(.)}$.

Let $(E, \|.\|_E)$ be a Banach space and E^* its dual space and (Ω, Σ, μ) be a measure space.

Definition 3. A function $f: \Omega \to E$ Bochner (or strongly) μ-measurable if there exists a sequence $\{f_n\}$ of simple functions $f_n: \Omega \to E$ such that $f_n(x) \to f(x)$ (in E) as n→∞ for almost all $x \in \Omega$ [8].

Theorem 4. A μ-measurable function $f: \Omega \to E$ is Bochner integrable if and only if $\int_{\mathbb{R}^n} \|f\|_E d\mu < \infty$ [8].

Definition 5. A function $F: \Sigma \to E$ is called a vector measure, if for all sequences $\{A_n\}$ of pairwise disjoint members of Σ such that $\bigcup_{n=1}^{\infty} A_n \in \Sigma$ and $F(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} F(A_n)$, where the series converges in the norm topology of E.

Let $F: \Sigma \to E$ be a vector measure. The variation of F is the function $\|F\|: \Sigma \to [0, \infty]$ defined by

$$\|F\|(A) = sup_\pi \sum_{B\in\pi}^{\infty} \|F(B)\|_E,$$

where the supremum is taken over all finite disjoint partitions π of A. If $\|F\|(\Omega) < \infty$, then F is called a measure of bounded variation [8].

Definition 6. A Banach space E has the Radon-Nikodym property (RNP) with respect to (Ω, Σ, μ) if for each vector measure $F: \Sigma \to E$ of bounded variation, which is absolutely continuous with respect to μ, there exists a function $g \in L^1(\Omega, \mathrm{E})$ such that

$$F(A) = \int_A g d\mu$$

for all $A \in \Sigma$ [8].

Definition 7. Let ϑ be a weight function and $1 < p^- \leq p(.) \leq p^+ < \infty$. The weighted variable exponent Bochner-Lebesgue space $L_\vartheta^{p(.)}(\mathbb{R}^n, E)$ stands for all (equivalence classes of) E -valued Bochner integrable functions f on $\mathbb{R}^n$ such that

$$L_\vartheta^{p(.)}(\mathbb{R}^n, E) = \{f: \|f\|_{p(.),\vartheta,E} < \infty\},$$

where $\|f\|_{p(.),\vartheta,E} = \left\| f\vartheta^{\frac{1}{p(.)}} \right\|_{p(.),E}$.

The following properties proved by Cheng and Xu [4];

(i) $f \in L_\vartheta^{p(.)}(\mathbb{R}^n, E) \Leftrightarrow \|f(.)\|_E^{p(.)} \in L_\vartheta^1(\mathbb{R}^n)$.

(ii) $L_\vartheta^{p(.)}(\mathbb{R}^n, E)$ is a generalization of the $L_\vartheta^{p(.)}(\mathbb{R}^n, E)$ spaces**.**

(iii) If $E = \mathbb{R}$ *or* $\mathbb{C}$, then $L_\vartheta^{p(.)}(\mathbb{R}^n, E) = L_\vartheta^{p(.)}(\mathbb{R}^n)$.

Theorem 8. $L_\vartheta^{p(.)}(\mathbb{R}^n, E)$ is a Banach space with respect to $\|.\|_{p(.),\vartheta,E}$.

Proof. Let $\{u_j\}$ be a Cauchy sequence in $L_\vartheta^{p(.)}(\mathbb{R}^n, E)$. Then, $\left\{u_j \vartheta^{\frac{1}{p(.)}}\right\}$ is a Cauchy sequence in the Banach space $L^{p(.)}(\mathbb{R}^n, E)$ in [7] due to

$$\|u_j - u_k\|_{p(.),\vartheta,E} = \left\| (u_j - u_k)\vartheta^{\frac{1}{p(.)}} \right\|_{p(.),E} \to 0,$$

so it converges to some u in $L^{p(.)}(\mathbb{R}^n, E)$. Consequently, $\{u_j\}$ converges to $u\vartheta^{-\frac{1}{p(.)}}$ in $L_\vartheta^{p(.)}(\mathbb{R}^n, E)$.

It is known that every weight function is equivalent to a continuous weight function. So we assume that the weight function ϑ is continuous (see [13]). Let $C_c(\mathbb{R}^n, E)$ denote the space of all E -valued continuous functions on $\mathbb{R}^n$ with compact support.

Theorem 9. $C_c(\mathbb{R}^n, E)$ is dense in $L_\vartheta^{p(.)}(\mathbb{R}^n, E)$.with respect to $\|.\|_{p(.),\vartheta,E}$.

Proof. Let $f \in L_\vartheta^{p(.)}(\mathbb{R}^n, E)$. Then $f\vartheta^{\frac{1}{p(.)}} \in L^{p(.)}(\mathbb{R}^n, E)$. Since $C_c(\mathbb{R}^n, E)$ is dense in $L^{p(.)}(\mathbb{R}^n, E)$, then there exists a function $f_c \in C_c(\mathbb{R}^n, E)$ such that

$$\left\| f - f_c\vartheta^{-\frac{1}{p(.)}} \right\|_{p(.),\vartheta,E} = \left\| f\vartheta^{\frac{1}{p(.)}} - f_c \right\|_{p(.),E} \to 0,$$

where $u\vartheta^{-\frac{1}{p(.)}} \in C_c(\mathbb{R}^n, E)$. This completes the proof.

III. Results

We define the space $A_{\vartheta_1,\vartheta_2}^{1,p(.)}(\mathbb{R}^n, E) = L_{\vartheta_1}^1(\mathbb{R}^n, E) \cap L_{\vartheta_2}^{p(.)}(\mathbb{R}^n, E)$ is a Banach space with the sum norm

$$\|f\|_{\vartheta_1,\vartheta_2,E}^{1,p(.)} = \|f\|_{1,\vartheta_1,E} + \|f\|_{p(.),\vartheta_2,E},$$

where $f \in A_{\vartheta_1,\vartheta_2}^{1,p(.)}(\mathbb{R}^n, E)$.

We say that $\vartheta_1 \prec \vartheta_2$ if and only if there exists a $C > 0$ such that $\vartheta_1(x) \leq C\vartheta_2(x)$ for all $x \in \mathbb{R}^n$. Two weight functions are called equivalent and written $\vartheta_1 \sim \vartheta_2$, if $\vartheta_1 \prec \vartheta_2$ and $\vartheta_2 \prec \vartheta_1$.

Proposition 10. Let $\vartheta_1, \vartheta_2, \vartheta_3$ and ϑ_4 be weight functions on $\mathbb{R}^n$. If $\vartheta_1 \prec \vartheta_3$ and $\vartheta_2 \prec \vartheta_4$, then the embedding $A_{\vartheta_3,\vartheta_4}^{1,p(.)}(\mathbb{R}^n, E) \hookrightarrow A_{\vartheta_1,\vartheta_2}^{1,p(.)}(\mathbb{R}^n, E)$ holds.

Proof. Let $f \in A_{\vartheta_3,\vartheta_4}^{1,p(.)}(\mathbb{R}^n, E)$. Since $\vartheta_1 \prec \vartheta_3$ and $\vartheta_2 \prec \vartheta_4$, then there exist C_1, $C_2 > 0$ such that $\vartheta_1(x) \leq C_1\vartheta_3(x)$ and $\vartheta_2(x) \leq C_2\vartheta_4(x)$ for all $x \in \mathbb{R}^n$. Hence we have $L_{\vartheta_3}^1(\mathbb{R}^n, E) \hookrightarrow L_{\vartheta_1}^1(\mathbb{R}^n, E)$ and $L_{\vartheta_4}^{p(.)}(\mathbb{R}^n, E) \hookrightarrow L_{\vartheta_2}^{p(.)}(\mathbb{R}^n, E)$. Using definition of the norm $\|.\|_{\vartheta_1,\vartheta_2,E}^{1,p(.)}$ we can write

$$\begin{aligned} \|f\|_{\vartheta_1,\vartheta_2,E}^{1,p(.)} &= \|f\|_{1,\vartheta_1,E} + \|f\|_{p(.),\vartheta_2,E} \\ &\leq C_1\|f\|_{1,\vartheta_3,E} + C_2\|f\|_{p(.),\vartheta_4,E} \\ &\leq C\|f\|_{\vartheta_3,\vartheta_4,E}^{1,p(.)}, \end{aligned}$$

where $C = maks\{C_1, C_2\}$.

Corollary 11. If $\vartheta_1 \sim \vartheta_3$ and $\vartheta_2 \sim \vartheta_4$, then the embedding $A_{\vartheta_1,\vartheta_2}^{1,p(.)}(\mathbb{R}^n, E) = A_{\vartheta_3,\vartheta_4}^{1,p(.)}(\mathbb{R}^n, E)$ holds.

Definition 12. Let $p_1(.)$ and $p_2(.)$ be exponents on $\mathbb{R}^n$. We say that $p_2(.)$ is non-weaker than $p_1(.)$ if and only if $\Phi_{p_1}(x, t) = t^{p_1(x)}$ is non-weaker than $\Phi_{p_2}(x, t) = t^{p_2(x)}$ in the sense of Musielak [12], i.e. there exist constants $K_1, K_2 > 0$ and $h \in L^1(\mathbb{R}^n), h \geq 0$, such that for a.e. $x \in \mathbb{R}^n$ and all $t \geq 0$

$$\Phi_{p_1}(x, t) \leq K_1\Phi_{p_2}(x, K_2t) + h(x).$$

We write $p_1 \preccurlyeq p_2$ [6].

Proposition 13. If $p_1 \preccurlyeq p_2$ and $\vartheta_1 \prec \vartheta_2$, then $L_{\vartheta_2}^{p_2(.)}(\mathbb{R}^n, E) \hookrightarrow L_{\vartheta_1}^{p_1(.)}(\mathbb{R}^n, E)$.

Proof. Since $p_1 \preccurlyeq p_2$, then $L_{\vartheta_2}^{p_2(.)}(\mathbb{R}^n, E) \hookrightarrow L_{\vartheta_2}^{p_1(.)}(\mathbb{R}^n, E)$ by Theorem 8.5 of [12]. Also by using Proposition 2.6 in [2], we have $L_{\vartheta_2}^{p_1(.)}(\mathbb{R}^n, E) \hookrightarrow L_{\vartheta_1}^{p_1(.)}(\mathbb{R}^n, E)$.

Corollary 14. If $\vartheta_1 \prec \vartheta_3, \vartheta_2 \prec \vartheta_4$ and $p_1 \preccurlyeq p_2$, then the embedding $A_{\vartheta_3,\vartheta_4}^{1,p_2(.)}(\mathbb{R}^n, E) \hookrightarrow A_{\vartheta_1,\vartheta_2}^{1,p_1(.)}(\mathbb{R}^n, E)$ holds.

Theorem 15. Let $p_1(.)$ and $p_2(.)$ be variable exponents satisfying $1 < p_1^- \leq p_1(.) \leq p_2(.) \leq p_2^+ < \infty$ and $\left\|\frac{\vartheta_1}{\vartheta_2}\right\|_{\frac{p_2(.)}{p_2(.)-p_1(.)},\vartheta_2} < \infty$. Then the embedding $L_{\vartheta_2}^{p_2(.)}(\mathbb{R}^n, E) \hookrightarrow L_{\vartheta_1}^{p_1(.)}(\mathbb{R}^n, E)$ holds.

Proof. Suppose that $f \in L^{p_2(.)}_{\vartheta_2}(\mathbb{R}^n, E)$. It is known that $L^{p_2(.)}_{\vartheta_2}(\mathbb{R}^n, E) \hookrightarrow L^{p_1(.)}_{\vartheta_1}(\mathbb{R}^n, E)$ with $\left\|\frac{\vartheta_1}{\vartheta_2}\right\|_{\frac{p_2(.)}{p_2(.)-p_1(.)},\vartheta_2} < \infty$ (Theorem 5.1, [9]). This completes the proof.

Theorem 16. Let $p_1(.)$ and $p_2(.)$ be variable exponents on $\mathbb{R}^n$. If the inclusion $A^{1,p_2(.)}_{\vartheta_3,\vartheta_4}(\mathbb{R}^n, E) \subset A^{1,p_1(.)}_{\vartheta_1,\vartheta_2}(\mathbb{R}^n, E)$ holds for the weights $\vartheta_1, \vartheta_2, \vartheta_3$ and ϑ_4 if and only if the embedding $A^{1,p_2(.)}_{\vartheta_3,\vartheta_4}(\mathbb{R}^n, E) \hookrightarrow A^{1,p_1(.)}_{\vartheta_1,\vartheta_2}(\mathbb{R}^n, E)$ is satisfied.

Proof. The sufficient condition of the theorem is clear by the definition of continuous embedding. Now, assume that the inclusion $A^{1,p_2(.)}_{\vartheta_3,\vartheta_4}(\mathbb{R}^n, E) \subset A^{1,p_1(.)}_{\vartheta_1,\vartheta_2}(\mathbb{R}^n, E)$ is valid. Moreover, we define the sum norm $|||f||| = \|f\|^{1,p_1(.)}_{\vartheta_1,\vartheta_2,E} + \|f\|^{1,p_2(.)}_{\vartheta_3,\vartheta_4,E}$. It is easy to see that $\left(A^{1,p_2(.)}_{\vartheta_3,\vartheta_4}(\mathbb{R}^n, E), |||.|||\right)$ is a Banach space. If we define the unit function I from $\left(A^{1,p_2(.)}_{\vartheta_3,\vartheta_4}(\mathbb{R}^n, E), |||.|||\right)$ to $\left(A^{1,p_2(.)}_{\vartheta_3,\vartheta_4}(\mathbb{R}^n, E), \|.\|^{1,p_2(.)}_{\vartheta_3,\vartheta_4,E}\right)$, then the function I is continuous. Because we can obtain the inequality $\|I(f)\|^{1,p_2(.)}_{\vartheta_3,\vartheta_4,E} = \|f\|^{1,p_2(.)}_{\vartheta_3,\vartheta_4,E} \le |||f|||$. By Banach's theorem I is a homeomorphism, see [3]. So the norms $|||.|||$ and $\|.\|^{1,p_2(.)}_{\vartheta_3,\vartheta_4,E}$ are equivalent. Thus, for every $f \in A^{1,p_2(.)}_{\vartheta_3,\vartheta_4}(\mathbb{R}^n, E)$ there exists a $k > 0$ such that

$$|||f||| \le k\|f\|^{1,p_2(.)}_{\vartheta_3,\vartheta_4,E}.$$

By the definition of the norm $|||.|||$ we have

$$\|f\|^{1,p_1(.)}_{\vartheta_1,\vartheta_2,E} \le |||f||| \le k\|f\|^{1,p_2(.)}_{\vartheta_3,\vartheta_4,E}.$$

Let $1 < s < \infty$. Consider the mapping Φ from $A^{s,p(.)}_{\vartheta_1,\vartheta_2}(\mathbb{R}^n, E)$ into $L^s_{\vartheta_1}(\mathbb{R}^n, E) \times L^{p(.)}_{\vartheta_2}(\mathbb{R}^n, E)$ defined by $\Phi(\mathrm{f}) = (f, f)$. This is a linear isometry of $A^{s,p(.)}_{\vartheta_1,\vartheta_2}(\mathbb{R}^n, E)$ into $L^s_{\vartheta_1}(\mathbb{R}^n, E) \times L^{p(.)}_{\vartheta_2}(\mathbb{R}^n, E)$ with the norm

$$|||(f, f)||| = \|f\|_{s,\vartheta_1,E} + \|f\|_{p(.),\vartheta_2,E}$$

for $f \in A^{s,p(.)}_{\vartheta_1,\vartheta_2}(\mathbb{R}^n, E)$. Hence it is easy to see that $A^{s,p(.)}_{\vartheta_1,\vartheta_2}(\mathbb{R}^n, E)$ is a closed subspace of the Banach space $L^s_{\vartheta_1}(\mathbb{R}^n, E) \times L^{p(.)}_{\vartheta_2}(\mathbb{R}^n, E)$. Let

$$H = \left\{(f, f) : f \in A^{s,p(.)}_{\vartheta_1,\vartheta_2}(\mathbb{R}^n, E)\right\}$$

and

$$K = \left\{\begin{matrix} (\varphi, \Psi) : (\varphi, \Psi) \in L^t_{\vartheta_1^*}(\mathbb{R}^n, E^*) \times L^{q(.)}_{\vartheta_2^*}(\mathbb{R}^n, E^*), \\ \int_{\mathbb{R}^n} f(x)\varphi(x)dx + \int_{\mathbb{R}^n} f(y)\Psi(y)dy = 0,\ for\ all\ (f, f) \in H \end{matrix}\right\}$$

where, $\frac{1}{\mathrm{s}} + \frac{1}{\mathrm{t}} = 1$ and $\frac{1}{\mathrm{p(.)}} + \frac{1}{\mathrm{q(.)}} = 1$.

The following Proposition is easily proved by Duality Theorem 1.7 in [11].

Proposition 17. The dual space $\left(A^{s,p(.)}_{\vartheta_1,\vartheta_2}(\mathbb{R}^n, E)\right)^*$ of $A^{s,p(.)}_{\vartheta_1,\vartheta_2}(\mathbb{R}^n, E)$ is isomorphic to $L^t_{\vartheta_1^*}(\mathbb{R}^n, E^*) \times L^{q(.)}_{\vartheta_2^*}(\mathbb{R}^n, E^*)$.

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