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FIXED POINT THEOREMS FOR MAPPINGS WHICH SATISFY (HRSC)-CONDITION

NURCAN BILGILI GUNGOR

Abstract. In 2008 Suzuki introduced C-condition for mappings which defined on a subset of a Banach space and presented fixed point theorems. And in 2010 Khan and Suzuki gave a reich type convergence theorem for generalized nonexpansive mappings in uniformly convex Banach space. Also in 2013 Karapınar introduced generalized C-conditions for mappings which defined on a subset of a Banach space and proved related fixed point theorems. And then in 2015 Thakur, Singh Thahur and Postolache proved fixed point theorems for mapping which satisfy RCSC-condition. In this paper fixed point theorems for mappings spaces which satisfy (HRSC)-condition are proved.

1. Introduction and Preliminaries

Let E be a Banach space and let K be a nonempty subset of E. A mapping T is called nonexpansive mapping if $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in K$.
We denote by $F(T)$ the set of fixed point of T.
A mapping T on a subset K of a Banach space E is called a quasi-nonexpansive mapping if $||Tx - z|| \leq ||x - z||$ for all $x \in K$ and $z \in F(T)$.
In 2008, Suzuki (see [9]) introduced the concept of generalized nonexpansive mappings (mappings is said to satisfy condition-(C)).

Definition 1.1. *([9]) Let T be a mapping on a subset K of a Banach space E. Then T is said to satisfy condition-(C) if*

$$\frac{1}{2}||x - Tx|| \leq ||x - y|| \text{ implies that } ||Tx - Ty|| \leq ||x - y|| \text{ for all } x, y \in K. \tag{1}$$

This condition is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. Indeed,

Proposition 1.1. *([9]) Every nonexpansive mapping satisfies condition-(C).*

Proposition 1.2. *([9]) Assume that a mapping T satisfies condition-(C) and has a fixed point. Then T is a quasi-nonexpansive mapping.*

If E is uniformly convex and K is bounded, closed and convex, then $F(T)$ is nonempty (See Browder [1], Göhde [2] and Kirk [6]). In 1979, the following interesting convergence theorem were given by Reich([8]).

Theorem 1.1. *([8]) Let E be a uniformly convex Banach space whose norm is Fréchet differentiable. Let T be a nonexpansive mapping on a bounded, closed and convex subset K of E. Define a sequence $\{x_n\}$ in K by $x_1 \in K$ and $x_{n+1} = \alpha_n T x_n + (1 - \alpha_n)x_n$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

In this this paper, respectively , the set of all positive integers and the set of all real numbers indicated by $\mathbb{N}$ and $\mathbb{R}$.

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Let E be a Banach space. if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $||x + y|| \leq 2 - \delta$ for all $x, y \in E$ with $||x|| = ||y|| = 1$ and $||x - y|| \geq \varepsilon$, then E is called uniformly convex Banach space. The following result was given by Kirk([6]).

Lemma 1.1. *([6]) Let T be a uniformly convex Banach space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in E which satisfy $\lim_{n\to\infty} ||x_n|| = 1$, $\lim_{n\to\infty} ||y_n|| = 1$ and $\lim_{n\to\infty} ||x_n + y_n|| = 2$. Then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.*

Khan and Suzuki gave the following Lemmas in [5] using the above Lemma,

Lemma 1.2. *([5]) Let E be a uniformly convex Banach space and let $\{u_n\}, \{v_n\}$ and $\{w_n\}$ be sequences in E. Let s and t be a real numbers with $s \in (0, \infty)$ and $t \in (0, 1)$. Assume that $\lim_{n\to\infty} ||u_n - v_n|| = s$, $\limsup_{n\to\infty} ||u_n - w_n|| \leq (1 - t)s$ and $\limsup_{n\to\infty} ||v_n - w_n|| \leq ts$. Then*

$$\lim_{n\to\infty} ||tu_n + (1 - t)v_n - w_n|| = 0. \tag{2}$$

A Banach space E is said to have the Kadec-Klee property if, for every sequence $\{x_n\}$ in E which converges weakly to a point $x \in E$ with $||x_n||$ converges to $||x||$, $\{x_n\}$ converges strongly to x. Uniformly convex Banach spaces and finite dimensional Banach spaces are typical examples. So, every L^p space with $1 < p < \infty$ is a uniformly convex Banach space whose dual has the Kadec-Klee property.

Lemma 1.3. *([5]) Let E be a reflexive Banach space whose dual has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and let $y, z \in E$ be weak subsequential limits of $\{x_n\}$. Assume that for every $t \in [0, 1]$, the limit of $\{||tx_n + (1 - t)y - z||\}$ exists. Then $y = z$.*

Suzuki gave the following Lemma in [9] .

Lemma 1.4. *Let $\{T\}$ be a mapping on a bounded and convex subset K of a Banach space E. Assume that T satisfies condition-(C). Define a sequence $\{x_n\}$ in K by $x_1 \in K$ and $x_{n+1} = \alpha Tx_n + (1 - \alpha)x_n$, where α is a real number belonging to $[\frac{1}{2}, 1)$. Then $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$.*

On the other hand Karapınar and Tas suggested new definitions in [3]. And then, Karapınar [4] suggested a new definition which is modification of Suzuki's C-condition:

Definition 1.2. *([4]) Let T be a mapping on a subset K of a Banach Space E. Then T is said to satisfy (for all $x, y \in K$)*
(i) Reich-Suzuki-(C) condition (in short , (RSC)-condition) if

$$\frac{1}{2} \| x - Tx \| \leq \| x - y \| \ \textit{ implies that} \tag{3}$$

$$\| Tx - Ty \| \leq \frac{1}{3}\{\| x - y \| + \| Tx - x \| + \| y - Ty \|\}, \tag{4}$$

(ii) Reich-Chatterjea-Suzuki-(C) condition (in short , (RCSC)-condition) if

$$\frac{1}{2} \| x - Tx \| \leq \| x - y \| \ \textit{ implies that} \tag{5}$$

$$\| Tx - Ty \| \leq \frac{1}{3}\{\| x - y \| + \| Tx - y \| + \| x - Ty \|\}, \tag{6}$$

(iii) Hardy-Rogers-Suzuki-(C) condition (in short , (HCSC)-condition) if

$$\frac{1}{2} \| x - Tx \| \leq \| x - y \| \ \textit{ implies that} \tag{7}$$

$$\| Tx - Ty \| \leq \frac{1}{5}\{\| x - y \| + \| Tx - x \| + \| y - Ty \| + \| Tx - y \| + \| x - Ty \|\}. \tag{8}$$

Also Karapınar proved some propositions in [4]:

Proposition 1.3. *([4]) If a mapping T satisfies (HRSC)-condition and has a fixed point, then it is quasi-nonexpansive mapping.*

Proposition 1.4. *([4]) Let T be a mapping on a closed subset K of a Banach space E. Assume that T satisfies (HRSC)-condition. Then $F(T)$ is closed. Moreover, E is strictly convex and K is convex, then $F(T)$ is also convex.*

Proposition 1.5. *([4]) Let T be a mapping on a subset K of a Banach space E and satisfy (HRSC)-condition. Then $\| x - Ty \| \leq 15 \| Tx - x \| + \| x - y \|$ holds for all $x, y \in K$.*

On the other hand, Thakur et al.[10] presented the following lemmas and theorems which are an extension of Lemmas and theorems in [5] to the case of mappings satisfying (RCSC)-condition:

Lemma 1.5. *([10]) Let T be a mapping on a bounded and convex subset K of a uniformly convex Banach space E. Assume that T satisfies Condition $(RCSC)$. Then for any $\varepsilon > 0$, there exists $\xi(\varepsilon) > 0$ such that for any $t \in [0,1]$ and for any $u, v \in K$ with $||Tu-u|| < \xi(\varepsilon), ||Tv-v|| < \xi(\varepsilon)$ we have*

$$||T(tu + (1-t)v) - (tu + (1-t)v|| < \varepsilon. \tag{9}$$

Proposition 1.6. *([10]) Let T be a mapping on a bounded and convex subset K of a uniformly convex Banach space E. Assume that T satisfies Condition $(RCSC)$. Then $I - T$ is demiclosed at zero. That is, if $\{x_n\}$ in K converges weakly to $z \in K$ and $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$ then Tz=z.*

Lemma 1.6. *([10]) Let T be a mapping on a compact convex subset K of a uniformly convex Banach space E and satisfy $(RCSC)$-condition. Define a sequence x_n in K by $x_1 \in K$ and $x_{n+1} = \lambda Tx_n + (1-\lambda)x_n$, for $n \in \mathbb{N}$, where λ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$ holds. Let $p, q \in F(T)$ and $t \in [0,1]$. Then the limit of $\{||tx_n + (1-t)p - q||\}$ exists.*

Theorem 1.2. *([10]) Let E be a uniformly convex Banach space whose dual has the Kadec-Klee property. Let T be a mapping on a bounded, closed, and convex subset K of E. Assume that T satisfies condition-$(RCSC)$. Define a sequence $\{x_n\}$ in K by $x_1 \in K$ and $x_{n+1} = \alpha Tx_n + (1-\alpha)x_n$ where α is a real number belonging to $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$ holds. Then $\{x_n\}$ converges weakly to a fixed point of T.*

2. Main Results

Now, we are ready to give and prove our main results. Firstly, using Proposition 1.5, we can prove the following Proposition:

Proposition 2.1. *Let T be a mapping on a subset K of a Banach space E and satisfies $(HRSC)$-condition. Then*

$$||y - Ty|| \leq 15||Tx - x|| + 2||x - y|| \text{ for all } x, y \in K. \tag{10}$$

Proof. If we use triangle inequality and Proposition 1.5, we will get

$$\begin{aligned} ||y - Ty|| &\leq ||y - x|| + ||x - Ty|| \\ &\leq ||y - x|| + 15||Tx - x|| + ||x - y|| \\ &\leq 15||Tx - x|| + 2||x - y||. \end{aligned} \tag{11}$$

□

Lemma 2.1. *Let T be a mapping on a bounded and convex subset K of a uniformly convex Banach space E. Assume that T satisfies $(HRSC)$-condition. Then for any $\varepsilon > 0$, there exists $\xi(\varepsilon) > 0$ such that for any $t \in [0,1]$ and for any $u, v \in K$ with $||Tu-u|| < \xi(\varepsilon), ||Tv-v|| < \xi(\varepsilon)$ we have*

$$||T(tu + (1-t)v) - (tu + (1-t)v|| < \varepsilon. \tag{12}$$

Proof. Arguing by contradiction, we assume that there exists $\varepsilon > 0$, sequences $\{u_n\}$ and $\{v_n\}$ in K and a sequence $\{t_n\}$ in $[0,1]$ such that $||Tu_n - u_n|| < \frac{1}{n}, ||Tv_n - v_n|| < \frac{1}{n}$ and $||T(t_n u_n + (1-t_n)v_n) - (t_n u_n + (1-t_n)v_n)|| \geq \varepsilon$. Denote $x_n = t_n u_n + (1-t_n)v_n$ and $w_n = Tx_n$. Using Proposition 2.1, we have that

$$\begin{aligned} 0 < \varepsilon \ &\leq \liminf_{n\to\infty} ||x_n - Tx_n|| \\ &\leq \liminf_{n\to\infty} (15||u_n - Tu_n|| + 2||x_n - u_n||) \\ &= 2\liminf_{n\to\infty} ||x_n - u_n||. \end{aligned} \tag{13}$$

Similarly we can show that $0 < \liminf_{n\to\infty} ||x_n - v_n||$. Thus, $0 < \liminf_{n\to\infty} ||u_n - v_n||$. Since K is bounded and

$$0 < \liminf_{n\to\infty} ||x_n - v_n|| = \liminf_{n\to\infty} t_n ||u_n - v_n|| \leq \liminf_{n\to\infty} t_n \times \sup_{n\in\mathbb{N}} ||u_n - v_n||, \tag{14}$$

we obtain that $0 < \liminf_{n\to\infty} t_n$. Similarly we can prove that $\limsup_{n\to\infty} t_n < 1$. Thus, without loss of generality, we may assume that $\{||u_n - v_n||\}$ and $\{t_n\}$ converge to some real numbers $s \in (0,\infty)$ and $t \in (0,1)$. Since $\lim_{n\to\infty} ||Tu_n - u_n|| = 0$ and $0 < \liminf_{n\to\infty} ||u_n - x_n||$, we have that $\frac{1}{2}||u_n - Tu_n|| \leq ||u_n - x_n||$ for sufficiently large $n \in \mathbb{N}$.
From Condition-$(HRSC)$, for sufficiently large $n \in \mathbb{N}$, we obtain that

$$|| Tu_n - Tx_n || \leq \frac{1}{5}\{|| u_n - x_n || + || Tu_n - u_n || + || x_n - Tx_n || + || Tu_n - x_n || + || u_n - Tx_n ||\}. \tag{15}$$

Similarly, for sufficiently large $n \in \mathbb{N}$, we can get

$$|| Tv_n - Tx_n || \leq \frac{1}{5}\{|| v_n - x_n || + || Tv_n - v_n || + || x_n - Tx_n || + || Tv_n - x_n || + || v_n - Tx_n ||\}. \tag{16}$$

Now, by using the triangular inequality and Proposition 1.5, we obtain

$$\begin{aligned} \limsup_{n\to\infty} || u_n - w_n || \ &\leq \limsup_{n\to\infty}(|| u_n - Tu_n || + || Tu_n - Tx_n ||) \\ &\leq \limsup_{n\to\infty}(|| u_n - Tu_n || + \tfrac{1}{5}\{|| u_n - x_n || + || Tu_n - u_n || + || x_n - Tx_n || \\ &\quad + || Tu_n - x_n || + || u_n - Tx_n ||\}) \\ &\leq \limsup_{n\to\infty}(|| u_n - Tu_n || + \tfrac{1}{5}\{5 || u_n - x_n || + 32 || Tu_n - u_n ||\}) \\ &= (1-t)d \end{aligned} \tag{17}$$

and

$$\begin{aligned} \limsup_{n\to\infty} || v_n - w_n || \ &\leq \limsup_{n\to\infty}(|| v_n - Tv_n || + || Tv_n - Tx_n ||) \\ &\leq \limsup_{n\to\infty}(|| v_n - Tv_n || + \tfrac{1}{5}\{|| v_n - x_n || + || Tv_n - v_n || + || x_n - Tx_n || \\ &\quad + || Tv_n - x_n || + || v_n - Tx_n ||\}) \\ &\leq \limsup_{n\to\infty}(|| u_n - Tu_n || + \tfrac{1}{5}\{5 || v_n - x_n || + 32 || Tv_n - v_n ||\}) \\ &= td \end{aligned} \tag{18}$$

Therefore from Lemma (1.2), we get that (2) holds. Hence we that

$$\begin{aligned} 0 < \varepsilon \ &\leq \lim_{n\to\infty} ||x_n - w_n|| \\ &\leq \lim_{n\to\infty} (||x_n - (tu_n + (1-t)v_n)|| + ||tu_n + (1-t)v_n - w_n||) = 0 \end{aligned} \tag{19}$$

and this is a contradiction. Thus, our assumption is wrong. ☐

Proposition 2.2. *Let T be a mapping on a bounded and convex subset K of a uniformly convex Banach space E. Assume that T satisfies $(HRSC)$-condition. Then $I-T$ is demiclosed at zero. That is, if $\{x_n\}$ in K converges weakly to $z \in K$ and $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$ then Tz=z.*

Proof. Let ξ be a function self mapping on $(0,\infty)$ which satisfies the conclusion of Lemma 2.1. We assume that $\{x_n\}$ converges weakly to $z \in K$ and $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$. Let $\varepsilon > 0$ is arbitrary. Define a strictly decreasing sequence $\{\varepsilon_n\}$ in $(0,\infty)$ by

$$\varepsilon_1 = \varepsilon \text{ and } \varepsilon_{n+1} = \min \frac{\{\varepsilon_n, \xi(\varepsilon_n)\}}{2}. \tag{20}$$

It is obvious that $\varepsilon_{n+1} < \xi(\varepsilon_n)$. Choose a subsequence $\{x_{f(n)}\}$ of $\{x_n\}$ such that $||x_{f(n)} - Tx_{f(n)}|| < \xi(\varepsilon_n)$. Since $\{x_{f(n)}\}$ converges weakly to z, z in the closed convex hull of $\{x_{f(n)} : n \in \mathbb{N}\}$. Thus there exists $y \in K$ and $v \in \mathbb{N}$ such that $||y - z|| < \varepsilon$ and y in the convex hull of $\{x_{f(n)} : n = 1, 2, ..., v\}$. From Lemma 2.1 we obtain that $||Ty - y|| < \varepsilon$. So we have from Proposition 2.1 that

$$||Tz - z|| \leq 15||Ty - y|| + 2||y - z|| \leq 7\varepsilon. \tag{21}$$

Since $\varepsilon > 0$ is arbitrary, we obtain $Tz = z$. □

Lemma 2.2. *Let T be a mapping on a compact convex subset K of a uniformly convex Banach space E and satisfies $(HRSC)$-condition. Define a sequence x_n in K by $x_1 \in K$ and $x_{n+1} = \lambda Tx_n + (1-\lambda)x_n$, for $n \in \mathbb{N}$, where λ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$ holds. Let $p, q \in F(T)$ and $t \in [0,1]$. Then the limit of $\{||tx_n + (1-t)p - q||\}$ exists.*

Proof. Define a mapping S on K by $Sx = \lambda Tx + (1-\lambda)x$. It is simple to see $F(S) = F(T)$ and S is quasi-nonexpansive. Since $x_{n+1} = S^n x_1$, $\{||x_n - q||\}$ is nonincreasing and hence converges. Also $\{||p - q||\}$ converges, obviously. Thus it is adequate to consider the case where $t \in (0,1)$. Denote $s = \lim_{n\to\infty} ||x_n - p||$. If $s = 0$, then the conclusion is obtained. So we assume that $s > 0$. Also we get

$$\begin{aligned}
\liminf_{m,n\to\infty} ||x_n - S^l(tx_m + (1-t)p)|| &\geq \liminf_{m,n\to\infty} (||x_n - p|| - ||p - S^l(tx_m + (1-t)p)||) \\
&\geq \lim_{m,n\to\infty} (||x_n - p|| - ||p - (tx_m + (1-t)p)||) \\
&= (1-t)s > 0
\end{aligned} \tag{22}$$

for all $l \in \mathbb{N} \cup \{0\}$, where S^0 is the identity mapping on K. Thus there exists $v \in \mathbb{N}$ such that

$$\frac{1}{2}||x_n - Tx_n|| \leq ||x_n - S^l(tx_m + (1-t)p)|| \tag{23}$$

for all $l \geq 0$ and $m, n \geq v$. Since T satisfies $(HRSC)$-condition, for all $l \geq 0$ and $m, n \geq v$, we get that

$$\begin{aligned}
||Tx_n - T \circ S^l(tx_m + (1-t)p)|| \leq \tfrac{1}{5}\{&||x_n - S^l(tx_m + (1-t)p)|| \\
&+ ||Tx_n - x_n|| + ||S^l(tx_m + (1-t)p) - T \circ S^l(tx_m + (1-t)p)|| \\
&+ ||Tx_n - S^l(tx_m + (1-t)p)|| + ||x_n - T \circ S^l(tx_m + (1-t)p)||]\}.
\end{aligned} \tag{24}$$

and so

$$\begin{aligned}
||x_{n+1} - S^{l+1}(tx_m + (1-t)p)|| &= ||Sx_n - S \circ S^l(tx_m + (1-t)p)|| \\
&\leq ||\lambda Tx_n + (1-\lambda)x_n - \lambda T \circ S^l(tx_m + (1-t)p) \\
&\quad -(1-\lambda)S^l(tx_m + (1-t)p)|| \\
&= ||\lambda(Tx_n - T \circ S^l(tx_m + (1-t)p)) \\
&\quad +(1-\lambda)(x_n - S^l(tx_m + (1-t)p))|| \\
&\leq \lambda||(Tx_n - T \circ S^l(tx_m + (1-t)p))|| \\
&\quad +(1-\lambda)||x_n - S^l(tx_m + (1-t)p)|| \\
&\leq \lambda\{\tfrac{1}{5}\{||x_n - S^l(tx_m + (1-t)p)|| \\
&\quad +||Tx_n - x_n|| + ||S^l(tx_m + (1-t)p) - T \circ S^l(tx_m + (1-t)p)|| \\
&\quad +||Tx_n - S^l(tx_m + (1-t)p)|| + ||x_n - T \circ S^l(tx_m + (1-t)p)||\}\} \\
&\quad +(1-\lambda)||x_n - S^l(tx_m + (1-t)p)|| \\
&\leq ||x_n - S^l(tx_m + (1-t)p)|| + \tfrac{32}{5}\lambda||Tx_n - x_n|
\end{aligned} \tag{25}$$

Define a function h from $\mathbb{N}$ into $[0, \infty)$ by

$$h(n) = ||tx_n + (1-t)p - q||. \tag{26}$$

And take two subsequences $\{f(n)\}$ and $\{g(n)\}$ of $\{n\}$ such that $v < f(1), f(n) < g(n)$ for each $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} h(f(n)) = \liminf_{n\to\infty} h(n), \quad \lim_{n\to\infty} h(g(n)) = \limsup_{n\to\infty} h(n). \tag{27}$$

We denote

$$u_n = x_{g(n)}, \quad v_n = p \text{ and } w_n = S^{g(n)-f(n)}(tx_{f(n)} + (1-t)p). \tag{28}$$

We have that

$$\lim_{n\to\infty} ||u_n - v_n|| = s, \tag{29}$$

$$\begin{aligned}
\limsup_{n\to\infty} ||u_n - w_n|| &= \limsup_{n\to\infty}(||x_{g(n)} - S^{g(n)-f(n)}(tx_{f(n)} + (1-t)p)|| \\
&\leq \limsup_{n\to\infty}(||x_{f(n)} - (tx_{f(n)} + (1-t)p)|| + \tfrac{32}{5}\lambda||Tx_n - x_n| \\
&= (1-t)\limsup_{n\to\infty}(||x_{f(n)} - p|| \\
&= (1-t)s
\end{aligned} \tag{30}$$

and so,

$$\limsup_{n\to\infty} ||v_n - w_n|| = ts. \tag{31}$$

Using (29),(30),(31) and Lemma 1.2, we obtain

$$\lim_{n\to\infty} ||tu_n + (1-t)v_n - w_n|| = 0. \tag{32}$$

Hence

$$\begin{aligned}
\limsup_{n\to\infty} h(n) &= \lim_{n\to\infty} h(g(n)) \\
&\leq \limsup_{n\to\infty}(||tx_{g(n)} + (1-t)p - S^{g(n)-f(n)}(tx_{f(n)} + (1-t)p)|| \\
&\quad +||S^{g(n)-f(n)}(tx_{f(n)} + (1-t)p) - q||) \\
&\leq \lim_{n\to\infty} h(f(n)) \\
&= \liminf_{n\to\infty} h(n).
\end{aligned} \tag{33}$$

So the limit of $\{h(n)\}$ exists.

□

Now the main theorem will be given.

Theorem 2.1. *Let E be a uniformly convex Banach space whose dual has the Kadec-Klee property. Let T be a mapping on a bounded, closed, and convex subset K of E. Assume that T satisfies $(HRSC)$ condition. Define a sequence $\{x_n\}$ in K by $x_1 \in K$ and $x_{n+1} = \alpha T x_n + (1-\alpha)x_n$ where α is a real number belonging to $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$ holds. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Let W be the set of all weak subsequential limits of $\{x_n\}$. From Proposition 2.2 we have that $W \subset F(T)$. So from Lemma 1.3 and Lemma 2.2, W consist of one element. Since E is reflexive, every subsequence of $\{x_n\}$ has a subsequence converges weakly to the unique element of W. Thus $\{x_n\}$ itself converges weakly to a fixed point of W. □

And so we get the result of Theorem 2.1 in the following.

Corollary 2.1. *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable. Let K, T, α and $\{x_n\}$ be as in Theorem 2.1. Then $\{x_n\}$ converges weakly to a fixed point of T.*

3. Aknowledgements

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Nurcan Bilgili Gungor,
Amasya University,
Department of Mathematics, Faculty of Science and Arts, 06500, Amasya, Turkey
E-mail address: bilgilinurcan@gmail.com, nurcan.bilgili@amasya.edu.tr