Statistical Equal Convergence on Weighted Spaces

Fadime Dirik\textsuperscript{1}\*\textsuperscript{,} Kamil Demirci\textsuperscript{1} and Sevda Yıldız\textsuperscript{1}

\textsuperscript{1}Department of Mathematics, Sinop University, Sinop, Turkey
\textsuperscript{*Corresponding author: fdirik@sinop.edu.tr

Abstract – The Korovkin theory has effective role in approximation theory. This theory is connected with the approximation to continuous functions by means of positive linear operators. Many mathematicians have investigated the Korovkin-type theorems by for a sequence of positive linear operators defined on different spaces by using various types of convergence. Firstly, A.D. Gadjiev has proved the weighted Korovkin type theorems, (Math. Zamet., 20 (1976) 781-786 (in Russian)). Later, these theorems are studied by many authors by means of different convergence methods. Recently, The definition of equal convergence for real functions was introduced by Császár and Laczkovich and they improved their investigations on this convergence. Later Das et al. introduced the ideas of \textit{I} and \textit{I}\textsuperscript{*}-equal convergence with the help of ideals by extending the equal convergence (Mat. Vesnik, vol:66, 2 (2014),165-177). In our work, we introduce a new type of statistical convergence on weighted spaces by using the notions of the equal convergence. We study its use in the Korovkin-type approximation theory. Then, we construct an example such that our new approximation result works but its classical and statistical cases do not work.

Keywords – Statistical equal convergence, Double sequences, Korovkin theorem, Equi-statistical convergence.

I. INTRODUCTION

Now we remind the concepts of weight functions and weight spaces. The function \(\rho: \mathbb{R} \to \mathbb{R}\) is called a weight function if it is continuous on \(\mathbb{R}\), \(\lim_{y \to \infty} \rho(y) = \infty\) and for all \(y \in \mathbb{R}\), \(\rho(y) \geq 1\). Then the corresponding space of real valued functions \(f\) defined on \(\mathbb{R}\) and satisfying \(|f(y)| \leq M \rho(y)\) (for all \(y \in \mathbb{R}\)) is called weighted space and denoted by \(B_\rho\), where \(M\) is a constant depending on the function \(f\). The weighted subspace \(C_\rho\) of \(B_\rho\) is given by

\[ C_\rho = \{ f \in B_\rho : f \text{ is continuous on } \mathbb{R} \}. \]

Then \(C_\rho\) and \(B_\rho\) are Banach spaces with the norm (see [1])

\[ \|g\|_\rho = \sup_{y \in \mathbb{R}} \left| \frac{f(y)}{\rho(y)} \right|. \]

Let \(\rho_1\) and \(\rho_2\) be two weight functions satisfying below conditions. Also assume that

\[ \lim_{y \to \infty} \frac{\rho_1(y)}{\rho_2(y)} = 0. \quad (1.1) \]

If \(T\) is a positive linear operator from \(C_{\rho_1}\) into \(B_{\rho_2}\), then the operator norm \(\|T\|_{C_{\rho_1} \to B_{\rho_2}}\) is given by

\[ \|T\|_{C_{\rho_1} \to B_{\rho_2}} = \sup_{\|f\|_{\rho_1} = 1} \|Tf\|_{\rho_2}. \]

Throughout this paper, we use the test functions \(F_j, j = 0, 1, 2\) defined by

\[ F_0(y) = \rho_1(y) \frac{1}{1+y^2}, F_1(y) = \frac{y \rho_1(y)}{1+y^2}, F_2(y) = \frac{y^2 \rho_1(y)}{1+y^2}. \]

Császár and Laczkovich was introduced the definition of equal convergence for real functions on a compact subset \(I\) of the real numbers \([2,3]\). Later, Das, Dutta and Pal introduced the ideas of \(I\) and \(I^*\)-equal with the help of ideals by extending the equal convergence [4].

Let’s remember this definition. Let \(\mathbb{N}\) be the set of natural numbers and let \(A \subset \mathbb{N}\). Also let

\[ A_n := \{ k \leq n : k \in A \} \]

and suppose that the symbol \(|A_n|\) denotes the cardinality of \(A_n\). Then the natural double density of \(A\) is defined by

\[ \delta(A) := \lim_{n \to \infty} \frac{1}{n} \left| k \leq n : k \in A \right| \]

provided that the limit exists. A given sequence \((y_n)\) is said to be statistically convergent to \(l\) if, for every \(\varepsilon > 0\), the following set:

\[ K = K(\varepsilon) := \{ n : |y_n - l| \geq \varepsilon \} \]

has natural density zero \([9]\), i.e., for every \(\varepsilon > 0\), we have

\[ \delta(K) := \lim_{n \to \infty} \frac{1}{n} \left| k \leq n : |y_n - l| \geq \varepsilon \right| = 0. \]

In this case, we show \(s^t - \lim_{n \to \infty} y_n = l\). It is known that, every convergent sequence is statistically convergent to same limit, but the converse is not true.

Let’s remember this definition.

Definition 1.1. [4] If there is a positive numbers sequence \((\varepsilon_n)\) with \(s^t - \lim_{n \to \infty} \varepsilon_n = 0\) such that for any \(y \in I\)
\[
\lim_{n \to \infty} \left| n \left[ f_n(y) - f(y) \right] \right| = 0,
\]
then \((f_n)\) is said to be statistical equal convergent to \(f\) on \(I\). In this case we write \(f_n \to f(eq-st)\) on \(I\).

Let \(f\) and \(f_n\) belong to \(C_{\rho_j}\).

**Definition 1.2.** \((f_n)\) is said to be statistical equal convergent to \(f\) on \(C_{\rho_j}\), if there is a positive numbers sequence \((\varepsilon_n)\) with \(st - \lim \varepsilon_n = 0\) such that for any \(y \in I\)

\[
\lim_{n \to \infty} \frac{1}{n} \left\{ \left[ n \left[ f_n(y) - f(y) \right] \right] \right\} = 0.
\]

Then, we write \(f_n \to f(eq-st)\) on \(I\).

Now we give an example.

**Example 1.1.** Let, for each \(y \in \mathbb{R}\), \(h(y) = 0\) and \((h_n)\) is a sequence of functions on \(R\) given by

\[
h_n(y) = \begin{cases} 
\frac{1}{1 + ny^2}, & \text{if } n \text{ is square}, \\
0, & \text{otherwise}.
\end{cases}
\]

Let \(\rho_j(y) = 1 + y^2\). Then every \(n \in \mathbb{N}\), \(h_n \in C_{\rho_j}\). Take \((\varepsilon_n)\) defined by

\[
\varepsilon_n = \begin{cases} 
n^2 + 1, & \text{if } n \text{ is square}, \\
\frac{1}{2n}, & \text{otherwise}.
\end{cases}
\]

Then it is easy to see that \(st - \lim \varepsilon_n = 0\). Also for any \(y \in \mathbb{R}\)

\[
\left\{ n : \left[ \frac{h_n(y) - h(y)}{\rho_j(y)} \right] \right\} = \emptyset.
\]

Therefore, we get \(h_n \to h(eq-st)\) on \(R\). But since \(\sup_{y \in \mathbb{R}} |h_n(y) - h(y)| = 1\) then \((h_n)\) is not statistical and classical uniform convergence to the function \(h = 0\) on \(\mathbb{R}\).

II. APPROXIMATION BY MEANS OF STATISTICAL EQUAL CONVERGENCE

In this section we prove a Korovkin type approximation theorem by means of the concept of statistical equal convergence.

Let \(T\) be a linear operator from \(C_{\rho_j}\) into \(B_{\rho_j}\). If \(h \geq 0\) implies \(T(h) \geq 0\), then we say that \(T\) is positive linear operator. Also, we denote the value of \(T(h)\) at a point \(y \in \mathbb{R}\) by \(T(h(u); y)\) or, briefly, \(T(h; y)\).

We need the following Lemmas to prove our main theorem.

**Lemma 2.1.** Let \((T_n)\) be a double sequence of positive linear operators from \(C_{\rho_j}\) into \(B_{\rho_j}\). If \(T_n(F_j) \to F_j(eq-st)\), \(j = 0, 1, 2\).

(2.1) \[
T_n(F_j) \to F_j(eq-st), j = 0, 1, 2.
\]

then, for any \(b > 0\) and for any \(|y| \leq b\), we have

\[
T_n(F) \to F(eq-st), \text{ for all } F \in C_{\rho_j}.
\]

**Proof.** Let \(F \in C_{\rho_j}\) and \(|y| \leq b\). Since \(F\) is continuous on \(\mathbb{R}\), we write that for every \(\varepsilon > 0\), there exists a number \(\delta > 0\) such that \(|F(u) - F(y)| < \varepsilon\) whenever \(|u - y| < \delta\). If \(|u - y| \geq \delta\), then we obtain

\[
|F(u) - F(y)| \leq 2M \rho_j(y) F_0(u) \left(1 + y^2\right) \\
\leq 4M \rho_j(y) F_0(u) \left(1 + \frac{1}{(u - y)^2}\right) \\
\leq S_{\rho_j}(y) F_0(u) \left(1 + \frac{1}{\delta^2}\right).
\]

where \(S_{\rho_j}(y) := 4M \rho_j(y) \left(1 + \frac{1}{\delta^2}\right)\). For all \(u \in \mathbb{R}\) and \(|y| \leq b\), we have

(2.2) \[
|F(u) - F(y)| \leq \varepsilon + S_{\rho_j}(y) F_0(u) \left(1 - \frac{1}{\delta^2}\right).
\]

Then, we write

\[
T_n(F; y) - F(y) \leq T_n(F(y); y) \\
+ \left| T_n(u) \right| \left| F_0(u) \left(1 - \frac{1}{\delta^2}\right) \right| \\
\leq T_n(\varepsilon + K_{\rho_j}(y) F_0(u) \left(1 - \frac{1}{\delta^2}\right) ; y) \\
+ |F(y)| \left| T_n(u) \left(1 - \frac{1}{\delta^2}\right) \right| \\
= \varepsilon T_n(1; y) + K_{\rho_j}(y) T_n(F_0(u) \left(1 - \frac{1}{\delta^2}\right) ; y) \\
+ |F(y)| \left| T_n(u) \left(1 - \frac{1}{\delta^2}\right) \right|.
\]

Hence, all \(y \in \mathbb{R}\) with \(|y| \leq b\), we have

\[
T_n(F; y) - F(y) \leq \varepsilon M_1 T_n(1; y) \\
+ M_2 \left| T_n(u) \left(1 - \frac{1}{\delta^2}\right) \right| \\
+ M_3 |T_n(u) - 1|.
\]

where
\( M_1 := M_1(b) := \sup_{|y| \leq b} \rho_1(y), \)
\( M_2 := M_2(b) := \sup_{|y| \leq b} S_\rho(y), \)
\( M_3 := M_3(b) := \sup_{|y| \leq b} |F(y)|. \)

For any \( y \in \mathbb{R} \) with \( |y| \leq b \) and \( b > 0 \)

\[
2 \sup_{|y| \leq b} |\rho_1(y)|, \quad \sup_{|y| \leq b} (y^2) \rho_1(y). \]

It follows from (2.3), that

\[
\left| T_n(F_0; y) - F_0(y) \right| \leq \frac{1}{F_0(y)} \left\{ \varepsilon T_n(1; y) + \sup_{|y| \leq b} |T_n(F_0; y) - F_0(y)| + \left| T_n(F_0; y) - F_0(y) \right|^2 \right\}. \]

Then, we have, for any \( y \in \mathbb{R} \) with \( |y| \leq b \) and \( b > 0 \) and for all \( n \in \mathbb{N} \), that

\[
\left| T_n(1; y) - 1 \right| \leq M_4 \left\{ \frac{T_n(F_0; y) - F_0(y)}{\rho_1(y)} + \varepsilon \frac{T_n(1; y)}{\rho_1(y)} + \sup_{|y| \leq b} S_\rho(y) T_n(F_0(y); y - y^2) \right\}.
\]

where
\[ M_4 := M_4(B) := \max \left\{ \sup_{|y| \leq b} \rho_1(y), \right\} \]

and considering the (2.3), (2.4) and (2.5), we obtain

\[
\left| T_n(F; y) - F(y) \right| \leq M \left\{ \varepsilon \frac{T_n(1; y)}{\rho_1(y)} + \sup_{|y| \leq b} |T_n(F_0; y) - F_0(y)| \right\}.
\]

where
\[ M := \max \{ M_1 + M_5 M_5, M_4 (M_1 + M_3 H_0) + M_3 M_5 \}. \]

By using (1.6), taking \( H = \max \left\{ M \sup_{|y| \leq b} \rho_1(y), M \right\} \) and since \( \varepsilon \) arbitrary, we get

\[
\left| T_n(F; y) - F(y) \right| \leq H \left( \sum_{t=0}^{\infty} \left| T_n(F_t; y) - F_n(y) \right| \right).
\]

for all \( n \in \mathbb{N} \) for some \( H > 0 \) independent of \( y \). Since \( T_n(F_j) \rightarrow F_j(eq-st) \) on \( \mathbb{R} \), there are positive number sequence \( \left( \varepsilon_{n,j} \right) \) with \( st - \lim_{n \rightarrow \infty} \varepsilon_{n,j} = 0 \) such that,

for any \( y \in \mathbb{R} \),

\[
\lim_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{t=0}^{\infty} \left| T_n(F_t; y) - F_n(y) \right| \right) \geq \varepsilon_{n,j} \]

\[
\in n \left( \left| \frac{T_n(F_t; y) - F_n(y)}{\rho_1(y)} \right| \geq 3\varepsilon_{n,j} \right).
\]

For any \( y \in \mathbb{R} \) with \( |y| \leq b \), from (2.7), we have

\[
\left( \left| \frac{T_n(F_t; y) - F_n(y)}{\rho_1(y)} \right| \geq 3\varepsilon_{n,j} \right).
\]
where \( \epsilon_n = \max \{ \epsilon_{n,t} : t = 0, 1, 2 \} \). Then using the hypothesis (2.11), we get, for any \( y \in \mathbb{R} \) with \( |y| \leq b \), for all \( F \in C_{\rho_1} \),
\[
T_n(F) \overset{\rho_1}{\rightarrow} F(eq-st)
\]
The proof of lemma is complete.

**Theorem 1.1.** Let \( \rho_1 \) and \( \rho_2 \) be as in Lemma 1.1. Let \( (T_n) \) is a sequence of positive linear operators acting \( C_{\rho_1} \) into \( B_{\rho_2} \). Then for all \( F \in C_{\rho_1} \),
\[
T_n(F) \overset{\rho_2}{\rightarrow} F(eq-st),
\]
on condition that
\[
T_n(F_j) \overset{\rho_1}{\rightarrow} F_j(eq-st), \quad j = 0, 1, 2
\]

**Proof.** We can show that the hypothesis (2.9) implies \( T_n(F_j;y) - F_j(y) \in B_{\rho_1} \) and therefore \( T_n(F;y) \in B_{\rho_1} \) for \( t = 0, 1, 2 \). Since \( \rho_1 = F_0 + F_2 \), we obtain \( T_n(\rho_1) \in B_{\rho_1} \) for each \( n \). If \( F \in C_{\rho_1} \) then, we can write \( T_n(F) \in B_{\rho_1} \).

Besides, we get, for \( H_i > 0 \),
\[
\|T_n\|_{S_{\rho_1} \rightarrow B_{\rho_1}} = \|T_n(\rho_1)\|_{\rho_1} = \sup_{y \in \mathbb{R}} \frac{T_n(\rho_1;y)}{\rho_1(n)} \leq H_i < \infty.
\]

Hence, we obtain for a given \( F \in C_{\rho_1} \), that
\[
\|T_n(F)\|_{\rho_1} \leq \|T_n\|_{S_{\rho_1} \rightarrow B_{\rho_1}} \|F\|_{\rho_1} \leq H_i \|F\|_{\rho_1}.
\]

For a given \( \varepsilon > 0 \), choose an \( b_0 > 0 \) such that \( \frac{\rho_1(n)}{\rho_2(n)} \leq \varepsilon \)

for every \( |y| \geq b_0 \). This is possible by (1.1). Using the fact, we obtain for \( F \in C_{\rho_1} \) and for any \( y \in \mathbb{R} \) with \( |y| \geq b_0 \), also by (2.10),
\[
\frac{T_n(F;y) - F(y)}{\rho_2(y)} = \frac{T_n(F;y) - F(y)}{\rho_1(y)} \frac{\rho_1(n)}{\rho_2(y)} \leq \varepsilon \frac{T_n(F;y) - F(y)}{\rho_1(y)}
\]
\[
\leq \varepsilon (\|T_n(F)\|_{\rho_1} + \|F\|_{\rho_1})
\]
\[
\leq \varepsilon \|F\|_{\rho_1} (H_i + 1)
\]
then, for \( F \in C_{\rho_1} \) and for any \( y \in \mathbb{R} \) with \( |y| \geq b_0 \)
\[
T_n(F) \overset{\rho_1}{\rightarrow} F(eq-st).
\]

Also, using the Lemma 1.1, for any \( b > 0 \) and for all \( F \in C_{\rho_1} \), we have for any \( y \in \mathbb{R} \) with \( |y| \leq b \),
\[
T_n(F) \overset{\rho_1}{\rightarrow} F(eq-st),
\]
then, we get from (2.11) and (2.12),
\[
T_n(F) \overset{\rho_1}{\rightarrow} F(eq-st) \quad \text{on} \quad \mathbb{R},
\]
the proof is completed.

III. Application

In this part, we give an example of a sequence of positive linear operators that satisfies the conditions of Theorem 1.1 but do not satisfy the conditions of Theorem 1.

**Example 3.1.** Consider the following linear positive operators given in [8] which is defined by:
\[
L_n(F;x) = \sum_{v=0}^{\infty} F\left(\frac{v}{\beta_n}\right)K_{n,v}(y)(\frac{-\alpha_n}{v})^v.
\]

Here \( (\alpha_n) \) and \( (\beta_n) \) be the real number sequences satisfying the followings:

(i) \( \lim_{n \to \infty} \beta_n = \infty \), (ii) \( \lim_{n \to \infty} \alpha_n = 0 \), (iii) \( \lim_{n \to \infty} \frac{\alpha_n}{\beta_n} = 1 \),

and \( K_{n,v}(y) \) is the functions satisfy the following conditions:

a) For any natural \( n, v = 0, 1, 2, \ldots \) and for any \( y \in [0, \infty) \)
\[
(-1)^v K_{n,v}(y) \geq 0
\]

b) For any \( y \in [0, \infty) \)
\[
\sum_{v=0}^{\infty} K_{n,v}(y)\left(\frac{-\alpha_n}{v}\right)^v = 1
\]

c) \( K_{n,v}(y) = -nK_{n,m,v+1}(y) \) for any \( y \in [0, \infty) \)

where \( n + m \) is a natural numbers and \( m \) is a constant independent of \( v \).

Then, using the operators \( L_n(F;x) \), introduce the following positive linear operators:
\[
T_n(F;y) = \left(1 + h_n(y)\right) L_n(F;y),
\]

\[
T_n(F) \overset{\rho_1}{\rightarrow} F(eq-st) \quad \text{on} \quad \mathbb{R},
\]
then, (3.3) holds true for \( t = 0 \). It is obvious that
\[
\left\| T_n (F_i ; y) - F_i (y) \right\| = \left\| (1 + h_n (y)) n \frac{\alpha_n}{\beta_n} y - y \right\| \\
= \left\| y \left( (1 + h_n (y)) n \frac{\alpha_n}{\beta_n} - 1 \right) \right\| \\
\leq \left\| y \right\| \left( \left| n \frac{\alpha_n}{\beta_n} - 1 \right| + h_n (y) n \frac{\alpha_n}{\beta_n} \right),
\]
then, by means of (iii) and \( h_n \xrightarrow{\rho_i} h(eq-st) \) on \( \mathbb{R} \),
\[
\lim_{n \to \infty} \left( n \frac{\alpha_n}{\beta_n} - 1 \right) = 0,
\]
\[
n \frac{\alpha_n}{\beta_n} h_n \xrightarrow{\rho_i} 0(eq-st) \quad \text{on } \mathbb{R}.
\]
Since \( \sup_{y \in (0, \infty)} \frac{|y|}{1 + y^2} < \infty \), we get
\[
T_n (F_i) \xrightarrow{\rho_i} F_i(eq-st) \quad \text{on } \mathbb{R},
\]
then, (3.3) holds true for \( t = 1 \). Finally, since
\[
\left\| T_n (F_i ; y) - F_i (y) \right\| \\
= \left\| (1 + h_n (y)) \left[ n (n + m) \frac{\alpha_n^2}{\beta_n^2} y^2 + n \frac{\alpha_n}{\beta_n^2} y \right] - y^2 \right\| \\
\leq \left\| y \right\| \left( \left| n (n + m) \frac{\alpha_n^2}{\beta_n^2} - 1 \right| + \left| h_n (y) n (n + m) \frac{\alpha_n^2}{\beta_n^2} \right| \right) \\
+ \left\| (1 + h_n (y)) n \frac{\alpha_n}{\beta_n^2} \right\|,
\]
because of (iii) and and \( h_n \xrightarrow{\rho_i} h(eq-st) \) on \( \mathbb{R} \), we can show that
\[
\lim_{n \to \infty} \left( n (n + m) \frac{\alpha_n^2}{\beta_n^2} - 1 \right) = 0,
\]
\[
n (n + m) \frac{\alpha_n^2}{\beta_n^2} h_n \xrightarrow{\rho_i} 0(eq-st) \quad \text{on } \mathbb{R},
\]
and since \( \lim_{\beta_n \to \infty} \frac{1}{\beta_n} = 0 \). From (i) and also using (iii), we get
\[
\lim n \frac{\alpha_n}{\beta_n} = 0,
\]
\[
n (n + m) \frac{\alpha_n}{\beta_n^2} h_n \xrightarrow{\rho_i} 0(eq-st) \quad \text{on } \mathbb{R}.
\]
Finally, since \( \sup_{y \in (0, \infty)} \frac{|y|}{1 + y^2} < \infty \), \( \sup_{y \in (0, \infty)} \frac{|y|}{1 + y^2} < \infty \) and from (3.4), (3.5), we obtain
\[
T_n (F_i) \xrightarrow{\rho_i} F_i(eq-st) \quad \text{on } \mathbb{R},
\]
Hence, our claim (3.3) holds true for each \( r = 0, 1, 2 \). \( (T_n) \) satisfies all hypothesis of Theorem 1.1 and we see that, for all \( F \in C_{\rho_i} \),
\[
T_n (F) \xrightarrow{\rho_i} F(eq-st) \quad \text{on } \mathbb{R}.
\]
However, since
\[
\left| T_n (F_i ; y) - F_i (y) \right| \xrightarrow{\rho_i} \left| (1 + h_n (y)) F_i (y) - F_i (y) \right| = h_n (y) \]