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A-Statistical Equal Approximation on Two Dimensional Weighted Spaces

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Abstract – Korovkin type approximation theorems have very important role in the approximation theory. Many mathematicians investigate and improve these type of approximation theorems for various operators defined on different spaces via several new convergence methods. The convergence of a sequence of positive linear operators defined on weighted space was first studied by Gadjiev [Theorems of Korovkin type, Math. Zametki 20(1976), 781-786]. Then, these results were improved by many authors for different type of convergence methods. Recently, some authors study Korovkin type theorems for two variables functions by means of single and double sequences on weighted spaces. In this paper, we prove a Korovkin type approximation theorem for the notion of statistical equal convergence for double sequences on two dimensional weighted spaces. Then, we construct an example such that our new approximation result works but its classical and statistical cases do not work. Also, we compute the rate of statistical equal convergence for double sequences on two dimensional weighted spaces.

Keywords – Double Sequences, Korovkin theorem, Statistical Equal Convergence, Weighted Spaces

I. Introduction and Preliminaries

We begin with some definitions and notations which we will use in the sequel.

A double sequence $x=(x_{ij})$ is said to be convergent in the Pringsheim sense if there exists a real number L such that for every $\varepsilon>0$ there exists a positive integer N, with $|x_{ij}-L|<\varepsilon$ whenever $i,j>N$. The number L is called the Pringsheim limit of x and denoted by $P-\lim_{i,j} x_{ij}$ (see [1]). More briefly, we will say that such an x is $P-$convergent to L. A double sequence is said to be bounded if there exists a positive number K such that $|x_{ij}|<K$ for all $(i,j)\in\mathbb{N}^2=\mathbb{N}\times\mathbb{N}$. Note that, in contrast to the case for single sequences, a convergent double sequence is not necessarily bounded.

Let now $A=(a_{mnij})$ be a four-dimensional summability method. For a given double sequence $x=(x_{ij})$, the $A-$transform of x, denoted by $Ax=((Ax)_{mn})$, is given by

$$(Ax)_{mn}=\sum_{i,j=1,1}^{\infty,\infty} a_{mnij}x_{ij},$$

provided the double series converges in the Pringsheim sense for $(m,n)\in\mathbb{N}^2$.

Recall that a four-dimensional matrix $A=(a_{mnij})$ is said to be $RH-$regular if it maps every bounded $P-$convergent sequence into a $P-$convergent sequence with the same $P-$limit. The Robison-Hamilton conditions (see also [2, 3]) state that a four-dimensional matrix $A=(a_{mnij})$ is $RH-$regular if and only if

(i) $P-\lim_{m,n} a_{mnij}=0$ for each i and j,

(ii) $P-\lim_{m,n}\sum_{i,j=1,1}^{\infty,\infty} a_{mnij}=1$,

(iii) $P-\lim_{m,n}\sum_{i=1}^{\infty}|a_{mnij}|=0$ for each j,

(iv) $P-\lim_{m,n}\sum_{j=1}^{\infty}|a_{mnij}|=0$ for each i,

(v) $\sum_{i,j=1,1}^{\infty,\infty}|a_{mnij}|$ is $P-$convergent for every $(m,n)\in\mathbb{N}^2$,

(vi) there exist finite positive integers B and C such that

$$\sum_{i,j>C}|a_{mnij}|<B$$

for every $(m,n)\in\mathbb{N}^2$.

Let $A=(a_{mnij})$ be a nonnegative $RH-$regular summability matrix. If $K\subset\mathbb{N}^2$, then the $A-$density of K is given by

$$\delta_A^2(K):=P-\lim_{m,n}\sum_{(i,j)\in K} a_{mnij},$$

provided that the limit on the right-hand side exists in the Pringsheim sense.

A real double sequence $x=(x_{ij})$ is said to be $A-$statistically convergent to L and denoted by $st_A^2-\lim_{i,j} x_{ij}=L$ if, for every $\varepsilon>0$,

$$P-\lim_{m,n}\sum_{(i,j)\in K(\varepsilon)} a_{mnij}=0,$$

where $K(\varepsilon)=\{(i,j)\in\mathbb{N}^2:|x_{ij}-L|\geq\varepsilon\}$ (see also [4, 5]). If we take $A=C(1,1)$, then $C(1,1)-$statistical convergence coincides with the notion of statistical convergence for double sequences (see also [6]), where $C(1,1)=(c_{mnij})$ is double Cesaro matrix, defined by $c_{mnij}=\frac{1}{mn}$, if $1\leq i\leq n$, $1\leq j\leq m$, and $c_{mnij}=0$, otherwise. We denote the set of all $A-$statistically convergent double sequences by st_A^2.

Now we recall the concepts of weight functions and two dimensional weighted space.

A real valued function ρ is called a weight function if it is continuous on $\mathbb{R}^2$ and for all $(t,u)\in\mathbb{R}^2$,

(1.1) $\rho(t,u)\geq 1$ and $\lim_{\sqrt{t^2+u^2}\to\infty}\rho(t,u)=\infty$.

Let B_ρ denote the weighted space which is the space of real valued functions f defined on $\mathbb{R}^2$ and satisfying $|f(t,u)|\leq M_f\rho(t,u)$ (for all $(t,u)\in\mathbb{R}^2$), where M_f is a constant depending on the function f. The weighted subspace C_ρ of B_ρ is given by

$$C_\rho:=\{f\in B_\rho: f \text{ is continuous on } \mathbb{R}^2\}.$$

The spaces B_ρ and C_ρ are Banach spaces with the norm

$$\|f\|_\rho:=\sup_{(t,u)\in\mathbb{R}^2}\frac{|f(t,u)|}{\rho(t,u)}, \text{ (see [7]).}$$

Let ρ_1 and ρ_2 be two weight functions satisfying (1.1). Assume also that the condition

(1.2) $$\lim_{\sqrt{t^2+u^2}\to\infty}\frac{\rho_1(t,u)}{\rho_2(t,u)}=0,$$

holds. If T is a positive linear operator from C_{ρ_1} into B_{ρ_2}, then we know that

$$\|T\|_{C_{\rho_1}\to B_{\rho_2}}:=\|T(\rho_1)\|_{\rho_2}.$$

Császár and Laczkovich were introduced the definition of equal convergence for real functions on a compact subset X of the real numbers and they improved their investigation on this convergence ([8], [9]). After that, Das, Dutta and Pal introduced the ideas of I and I^*-equal with the help of ideals by extending the equal convergence([10]). More recently, Okçu Şahin and Dirik ([11]) have been introduced the concept of statistical equal convergence of double function sequences. In this work, we prove a Korovkin theorem for $A-$statistical equal convergence on two dimensional weighted spaces. We also study the rate of $A-$statistical equal convergence by using the weighted modulus of continuity and afterwards, displaying an example, it is shown that our new result is stronger than classical and statistical cases.

First, we introduce the concept of $A-$statistical equal convergence for double function sequences.

Let f and f_n belong to C_ρ.

Definition 1.1. Let $A=(a_{mnij})$ be a non-negative $RH-$regular summability matrix. (f_{ij}) is said to be $A-$statistically equal convergent to f if there is a positive numbers sequence (ε_{ij}) with $st_A^2-\lim_{i,j}\varepsilon_{ij}=0$ such that for any $(t,u)\in\mathbb{R}^2$,

$$\delta_A^2\left(\left\{(i,j)\in\mathbb{N}^2:\left|\frac{f_{ij}(t,u)-f(t,u)}{\rho(t,u)}\right|\geq\varepsilon_{ij}\right\}\right)=0.$$

In this case we write $st_A^2-\lim_{i,j} f_{ij}=f\,(\rho-equal)$ on $\mathbb{R}^2$.

Now, we give an example which satisfies that $A-$statistical equal convergence is stronger than $A-$statistical uniform convergence for double function sequence.

Example 1.1. Let $g_{ij}:\mathbb{R}^2\to\mathbb{R}$, for each $(t,u)\in\mathbb{R}^2$, $g_{ij}(t,u)$ is given by

(1.3) $$g_{ij}(t,u)=\begin{cases}\dfrac{2}{3+it^2+j^2u^4}, & (i,j) \text{ is square,}\\ 0, & \textit{otherwise.}\end{cases}$$

Let $\rho(t,u)=1+t^2+u^2$ and $A=C(1,1)$. Then $g_{ij}\in C_\rho$ ($\forall i,j\in\mathbb{N}$). Take (ε_{ij}) defined by

$$\varepsilon_{ij}=\begin{cases}\dfrac{ij+1}{3}, & (i,j) \text{ is square,}\\ \dfrac{1}{i+j}, & \textit{otherwise.}\end{cases}$$

Then it is easy to see that $st^2-\lim_{i,j}\varepsilon_{ij}=0$. Also, for $g=0$ and any $(t,u)\in\mathbb{R}^2$, $\left\{(i,j)\in\mathbb{N}^2:\left|\frac{g_{ij}(t,u)}{\rho(t,u)}\right|\geq\varepsilon_{ij}\right\}=\{(1,1)\}$. Therefore, we get $st^2-\lim_{i,j} g_{ij}=0\,(\rho-equal)$ on $\mathbb{R}^2$. But, since $\sup_{(t,u)\in\mathbb{R}^2}\left|\frac{g_{ij}(t,u)}{\rho(t,u)}\right|=\frac{2}{3}$, then (g_{ij}) is not statistically (or ordinary) uniformly and also, (g_{ij}) is not equally convergent to the function $g=0$ on $\mathbb{R}^2$.

II. Approximation via A-statistical equal convergence

The Korovkin theory has been widely studied in the literature ([12]). This theory is connected with the approximation to continuous functions by means of positive linear operators (see, for instance, [13-15]). In this section we give Korovkin theorems for $A-$statistical equal convergence of double sequences of positive linear operators from C_{ρ_1} into B_{ρ_2} and we obtain the rate of $A-$statistical equal convergence. We give the proofs in consideration of revised proofs in [16].

First of all, we give the following.

Let (T_{ij}) be a sequence of positive linear operators from C_{ρ_1} into B_{ρ_2}. Since $\rho_1 \in C_{\rho_1}$, then $T_{ij}(\rho_1) \in B_{\rho_2}$ and therefore $\|T_{ij}(\rho_1)\|_{\rho_2} < \infty$. Furthermore, $\|T\|_{C_{\rho_1} \to B_{\rho_2}} < \infty$ which implies the uniform boundedness of T_{ij}.

Now, we recall the following Korovkin type approximation theorem on two-dimensional weighted space for double sequences before giving our main theorem.

Throughout this paper, let the test functions F_r ($r = 0,1,2,3$) defined by

$$F_0(t,u) = \frac{\rho_1(t,u)}{1+t^2+u^2},\ F_1(t,u) = \frac{t\rho_1(t,u)}{1+t^2+u^2},$$

$$F_2(t,u) = \frac{u\rho_1(t,u)}{1+t^2+u^2},\ F_3(t,u) = \frac{(t^2+u^2)\rho_1(t,u)}{1+t^2+u^2}.$$

Theorem 2.1. [17] Assume that the functions ρ_1 and ρ_2 are weight functions satisfying (1.2) and let (T_{ij}) is a double sequence of positive linear operators from C_{ρ_1} into B_{ρ_2}. Then, for all $f \in C_{\rho_1}$,

$$P-\lim_{i,j} \|T_{ij}(f) - f\|_{\rho_2} = 0$$

if

$$P-\lim_{i,j} \|T_{ij}(F_r) - F_r\|_{\rho_1} = 0,\ r = 0,1,2,3.$$

Theorem 2.2. [17] Let ρ_1 and ρ_2 are weight functions satisfying (1.2). Assume that (T_{ij}) is a double sequence of positive linear operators from C_{ρ_1} into B_{ρ_2}. Then, for all $f \in C_{\rho_1}$,

$$st^2-\lim_{i,j} \|T_{ij}(f) - f\|_{\rho_2} = 0$$

if

$$st^2-\lim_{i,j} \|T_{ij}(F_r) - F_r\|_{\rho_1} = 0,\ r = 0,1,2,3.$$

Now, we begin with following lemma.

Lemma 2.1. Let $A = (a_{mnij})$ be a non-negative $RH-$regular summability matrix and let (T_{ij}) be a double sequence of positive linear operators from C_{ρ_1} into B_{ρ_2} where ρ_1 and ρ_2 are weight functions satisfying the condition (1.2). If

(2.1) $st_A^2 - \lim_{i,j} T_{ij}(F_r) = F_r(\rho_1 - equal)$ on $\mathbb{R}^2$,

$r = 0,1,2,3$. Then, for any $a > 0$ and for all $f \in C_{\rho_1}$, we have for any $(t,u) \in \mathbb{R}^2$ with $\sqrt{t^2+u^2} \le a$

(2.2) $$st_A^2 - \lim_{i,j} T_{ij}(f) = f(\rho_2 - equal).$$

Proof. Let $f \in C_{\rho_1}$ and $\sqrt{t^2+u^2} \le a$. Since f is continuous on $\mathbb{R}^2$, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(s,v) - f(t,u)| < \varepsilon$ with $|s-t| < \delta$ and $|v-u| < \delta$. When $|s-t| \ge \delta$ or $|v-u| \ge \delta$, we have

$$\begin{aligned}
|f(s,v) - f(t,u)| &< 2M_f \rho_1(t,u)\rho_1(s,v) \\
&= 2M_f \rho_1(t,u) F_0(s,v)\left(1+s^2+v^2\right) \\
&\le 4M_f \rho_1(t,u) F_0(s,v)\left(1+t^2+u^2+(s-t)^2+(v-u)^2\right) \\
&= 4M_f \rho_1(t,u) F_0(s,v)\left[(s-t)^2+(v-u)^2\right] \left(\frac{1+t^2+u^2}{(s-t)^2+(v-u)^2}+1\right) \\
&\le K_{\rho_1}(t,u)\left[(s-t)^2+(v-u)^2\right] F_0(s,v),
\end{aligned}$$

where $K_{\rho_1}(t,u) := 4M_f \rho_1(t,u)\left\{1+\frac{1+t^2+u^2}{\delta^2}\right\}$. So, for all $(s,v) \in \mathbb{R}^2$ and $\sqrt{t^2+u^2} \le a$, we see that

(2.3) $$|f(s,v) - f(t,u)| < \varepsilon + K_{\rho_1}(t,u)\left[(s-t)^2+(v-u)^2\right] F_0(s,v).$$

Then, we can write

$$\begin{aligned}
|T_{ij}(f;t,u) - f(t,u)| &\le T_{ij}\left(|f(s,v) - f(t,u)|;t,u\right) + |f(t,u)|\,|T_{ij}(1;t,u) - 1| \\
&\le T_{ij}\left(\varepsilon + K_{\rho_1}(t,u)\left[(s-t)^2+(v-u)^2\right] F_0(s,v);t,u\right) + |f(t,u)|\,|T_{ij}(1;t,u) - 1| \\
&= \varepsilon T_{ij}(1;t,u) + |f(t,u)|\,|T_{ij}(1;t,u) - 1| \\
&\quad + K_{\rho_1}(t,u) T_{ij}\left(F_0(s,v)\left[(s-t)^2+(v-u)^2\right];t,u\right).
\end{aligned}$$

Hence, for any $(t,u) \in \mathbb{R}^2$ with $\sqrt{t^2+u^2} \le a$

$$\left|T_{ij}\left(f;t,u\right)-f\left(t,u\right)\right|$$

(2.4) $$\leq \varepsilon H_1 \frac{T_{ij}\left(1;t,u\right)}{\rho_1\left(t,u\right)}+H_2\left|T_{ij}\left(1;t,u\right)-1\right|$$

$$+H_3T_{ij}\left(F_0\left(s,v\right)\left[\left(s-t\right)^2+\left(v-u\right)^2\right];t,u\right)$$

where $H_1:=H_1\left(a\right):=\sup_{\sqrt{t^2+u^2}\leq a}\rho_1\left(t,u\right)$,

$H_2:=H_2\left(a\right):=\sup_{\sqrt{t^2+u^2}\leq a}\left|f\left(t,u\right)\right|$ and

$H_3:=H_3\left(a\right):=\sup_{\sqrt{t^2+u^2}\leq a}K_{\rho_1}\left(t,u\right)$. For any

$\left(t,u\right)\in\mathbb{R}^2$ with $\sqrt{t^2+u^2}\leq a$ and $a\in\mathbb{R}$,

(2.5)

$$T_{ij}\left(F_0\left(s,v\right)\left[\left(s-t\right)^2+\left(v-u\right)^2\right];t,u\right)$$

$$\leq\left\{\left|T_{ij}\left(F_3;t,u\right)-F_3\left(t,u\right)\right|+2\left|t\right|\left|T_{ij}\left(F_1;t,u\right)-F_1\left(t,u\right)\right|\right.$$

$$+2\left|u\right|\left|T_{ij}\left(F_2;t,u\right)-F_2\left(t,u\right)\right|$$

$$\left.+\left(t^2+u^2\right)\left|T_{ij}\left(F_0;t,u\right)-F_0\left(t,u\right)\right|\right\}$$

$$\leq H_4\left\{\frac{\left|T_{ij}\left(F_0;t,u\right)-F_0\left(t,u\right)\right|}{\rho_1\left(t,u\right)}+\frac{\left|T_{ij}\left(F_1;t,u\right)-F_1\left(t,u\right)\right|}{\rho_1\left(t,u\right)}\right.$$

$$\left.+\frac{\left|T_{ij}\left(F_2;t,u\right)-F_2\left(t,u\right)\right|}{\rho_1\left(t,u\right)}+\frac{\left|T_{ij}\left(F_3;t,u\right)-F_3\left(t,u\right)\right|}{\rho_1\left(t,u\right)}\right\}$$

where $$H_4:=H_4\left(a\right)=\max\left\{\sup_{\sqrt{t^2+u^2}\leq a}\rho_1\left(t,u\right),\right.$$

$$2\sup_{\sqrt{t^2+u^2}\leq a}\left|t\right|\rho_1\left(t,u\right),2\sup_{\sqrt{t^2+u^2}\leq a}\left|u\right|\rho_1\left(t,u\right)$$

$$\left.\sup_{\sqrt{t^2+u^2}\leq a}\left(t^2+u^2\right)\rho_1\left(t,u\right)\right\}.$$

Since $F_0\in C_{\rho_1}$ and

$$F_0\left(t,u\right)\left|T_{ij}\left(1;t,u\right)-1\right|$$

$$\leq T_{ij}\left(\left|F_0\left(s,v\right)-F_0\left(t,u\right)\right|;t,u\right)$$

$$+\left|T_{ij}\left(F_0;t,u\right)-F_0\left(t,u\right)\right|,$$

It follows from (2.3), that

$$\left|T_{ij}\left(1;t,u\right)-1\right|\leq\frac{1}{F_0\left(t,u\right)}\left\{\varepsilon T_{ij}\left(1;t,u\right)\right.$$

$$+\left|T_{ij}\left(F_0;t,u\right)-F_0\left(t,u\right)\right|$$

$$\left.+K_{\rho_1}\left(t,u\right)T_{ij}\left(F_0\left(s,v\right)\left[\left(s-t\right)^2+\left(v-u\right)^2\right];t,u\right)\right\}.$$

Hence, we have, for any $\left(t,u\right)\in\mathbb{R}^2$ with $\sqrt{t^2+u^2}\leq a$ and $a\in\mathbb{R}$ and for all $i,j\in\mathbb{N}$, that

$$\left|T_{ij}\left(1;t,u\right)-1\right|$$

(2.6) $$\leq H_5\left\{\frac{\left|T_{ij}\left(F_0;t,u\right)-F_0\left(t,u\right)\right|}{\rho_1\left(t,u\right)}+\varepsilon\frac{T_{ij}\left(1;t,u\right)}{\rho_1\left(t,u\right)}\right\}$$

$$+H_6T_{ij}\left(F_0\left(s,v\right)\left[\left(s-t\right)^2+\left(v-u\right)^2\right];t,u\right)$$

where

$H_5:=H_5\left(a\right):=\sup_{\sqrt{t^2+u^2}\leq a}\frac{\rho_1\left(t,u\right)}{F_0\left(t,u\right)}$ and

$H_6:=H_6\left(a\right):=\sup_{\sqrt{t^2+u^2}\leq a}\frac{K_{\rho_1}\left(t,u\right)}{F_0\left(t,u\right)}$. Also, for each $\left(i,j\right)\in\mathbb{N}^2$,

$$\sup_{\sqrt{t^2+u^2}\leq a}\frac{T_{ij}\left(1;t,u\right)}{\rho_1\left(t,u\right)}\leq\sup_{\sqrt{t^2+u^2}\leq a}\frac{T_{ij}\left(\rho_1;t,u\right)}{\rho_1\left(t,u\right)}$$

$$\leq\sup_{\sqrt{t^2+u^2}\leq a}\frac{\left|T_{ij}\left(\rho_1;t,u\right)-\rho_1\left(t,u\right)\right|}{\rho_1\left(t,u\right)}+1$$

$$\leq\sup_{\sqrt{t^2+u^2}\leq a}\frac{\left|T_{ij}\left(F_3;t,u\right)-F_3\left(t,u\right)\right|}{\rho_1\left(t,u\right)}$$

$$+\sup_{\sqrt{t^2+u^2}\leq a}\frac{\left|T_{ij}\left(F_0;t,u\right)-F_0\left(t,u\right)\right|}{\rho_1\left(t,u\right)}+1,$$

From which

(2.7) $$\sup_{\sqrt{t^2+u^2}\leq a}\frac{T_{ij}\left(1;t,u\right)}{\rho_1\left(t,u\right)}<\infty$$

follows and considering the (2.4), (2.5) and (2.6), we have

$$\left|T_{ij}\left(f;t,u\right)-f\left(t,u\right)\right|$$

$$\leq H\left\{\varepsilon\frac{T_{ij}\left(1;t,u\right)}{\rho_1\left(t,u\right)}+\frac{\left|T_{ij}\left(F_0;t,u\right)-F_0\left(t,u\right)\right|}{\rho_1\left(t,u\right)}\right.$$

$$+\frac{\left|T_{ij}\left(F_1;t,u\right)-F_1\left(t,u\right)\right|}{\rho_1\left(t,u\right)}+\frac{\left|T_{ij}\left(F_2;t,u\right)-F_2\left(t,u\right)\right|}{\rho_1\left(t,u\right)}$$

$$\left.+\frac{\left|T_{ij}\left(F_3;t,u\right)-F_3\left(t,u\right)\right|}{\rho_1\left(t,u\right)}\right\},$$

where

$$H:=\max\left\{H_1+H_3H_5,H_4\left(H_2+H_3H_6\right)+H_3H_5\right\}.$$

By using (2.7), taking

$M=\max\left\{H\sup_{\sqrt{t^2+u^2}\leq a}\frac{T_{ij}\left(1;t,u\right)}{\rho_1\left(t,u\right)},H\right\}$ and since ε arbitrary, we get

(2.8)

$$\frac{\left|T_{ij}\left(f;t,u\right)-f\left(t,u\right)\right|}{\rho_2\left(t,u\right)}\leq M\left\{\sum_{r=0}^{3}\frac{\left|T_{ij}\left(F_r;t,u\right)-F_r\left(t,u\right)\right|}{\rho_1\left(t,u\right)}\right\}$$

for all $i,j\in\mathbb{N}$ for some $M>0$ independent of (t,u). Since $st_A^2-\lim_{i,j}T_{ij}(F_r)=F_r(\rho_1-equal)$ on $\mathbb{R}^2$, $r=0,1,2,3$, there are positive number sequence $(\varepsilon_{ij,r})$ with $st_A^2-\lim_{i,j}\varepsilon_{ij,r}=0$ such that, for any $(t,u)\in\mathbb{R}^2$,

(2.9) $P-\lim_{m,n}\sum_{(i,j)\in\psi_{ij,r}(t,u,\varepsilon_{ij,r})}a_{mnij}=0$,

where

$$\psi_{ij,r}(t,u,\varepsilon_{ij,r}):=\left\{(i,j)\in\mathbb{N}^2:\left|\frac{T_{ij}(F_r;t,u)-F_r(t,u)}{\rho_1(t,u)}\right|\geq\varepsilon_{ij,r}\right\}$$

$r=0,1,2,3$. Then, for any $(t,u)\in\mathbb{R}^2$ with $\sqrt{t^2+u^2}\leq a$,

$$\psi_{ij}(t,u,\varepsilon_{ij}):=\left\{(i,j)\in\mathbb{N}^2:\left|\frac{T_{ij}(f;t,u)-f(t,u)}{\rho_2(t,u)}\right|\geq 4M\varepsilon_{ij}\right\}$$

where $\varepsilon_{ij}=\max\{\varepsilon_{ij,r}:r=0,1,2,3\}$. It follows from (2.8) that $\psi_{ij}(t,u,\varepsilon_{ij})\subseteq\bigcup_{r=0}^{3}\psi_{ij,r}(t,u,\varepsilon_{ij,r})$ and hence

$$\sum_{(i,j)\in\psi_{ij}(t,u,\varepsilon_{ij})}a_{mnij}\leq\sum_{r=0}^{3}\sum_{(i,j)\in\psi_{ij,r}(t,u,\varepsilon_{ij,r})}a_{mnij}.$$

Then using the hypothesis (2.1), we get, for any $(t,u)\in\mathbb{R}^2$ with $\sqrt{t^2+u^2}\leq a$,

$$st_A^2-\lim_{i,j}T_{ij}(f)=f(\rho_2-equal).$$

This completes the proof of the theorem.

Now, we can give our main Korovkin type approximation theorem.

Theorem 2.3. Let $A=(a_{mnij})$ be a non-negative $RH-$regular summability matrix and let ρ_1 and ρ_2 be as in Lemma 2.1. Suppose that (T_{ij}) is a double sequence of positive linear operators from C_{ρ_1} into B_{ρ_2}. Then for all $f\in C_{\rho_1}$,

(2.10) $st_A^2-\lim_{i,j}T_{ij}(f)=f(\rho_2-equal)$ on $\mathbb{R}^2$,

provided that

(2.11) $st_A^2-\lim_{i,j}T_{ij}(F_r)=F_r(\rho_1-equal)$ on $\mathbb{R}^2$,

$r=0,1,2,3$.

Proof. Observe that the hypothesis (2.11) implies $T_{ij}(F_r;t,u)-F_r(t,u)\in B_{\rho_1}$ and hence $T_{ij}(F_r;t,u)\in B_{\rho_1}$ for $r=0,1,2,3$. Since $\rho_1=F_0+F_3$, we also get $T_{ij}(\rho_1)\in B_{\rho_1}$ for each i,j. If $f\in C_{\rho_1}$ then, we obtain $T_{ij}(f)\in B_{\rho_1}$. Furthermore, we get, for $M_1>0$,

$$\|T_{ij}\|_{C_{\rho_1}\to B_{\rho_1}}=\|T_{ij}(\rho_1)\|_{\rho_1}=\sup_{(t,u)\in\mathbb{R}^2}\frac{T_{ij}(\rho_1;t,u)}{\rho_1(t,u)}\leq M_1<\infty.$$

Therefore we may write for a given $f\in C_{\rho_1}$, that

(2.12) $\|T_{ij}(f)\|_{\rho_1}\leq\|T_{ij}\|_{C_{\rho_1}\to B_{\rho_1}}\|f\|_{\rho_1}\leq M_1\|f\|_{\rho_1}$.

Now for a given $\varepsilon>0$, pick an $a_0>0$ such that $\frac{\rho_1(t,u)}{\rho_2(t,u)}\leq\varepsilon$ for every $\sqrt{t^2+u^2}\geq a_0$. This is possible by (1.2). Using the fact, we may write for $f\in C_{\rho_1}$ and for any $(t,u)\in\mathbb{R}^2$ with $\sqrt{t^2+u^2}\geq a_0$, also by (2.12),

$$\frac{|T_{ij}(f;t,u)-f(t,u)|}{\rho_2(t,u)}=\frac{|T_{ij}(f;t,u)-f(t,u)|}{\rho_2(t,u)}\frac{\rho_1(t,u)}{\rho_1(t,u)}$$
$$\leq\varepsilon\frac{|T_{ij}(f;t,u)-f(t,u)|}{\rho_1(t,u)}\leq\varepsilon\left(\|T_{ij}(f)\|_{\rho_1}+\|f\|_{\rho_1}\right)$$
$$\leq\varepsilon\|f\|_{\rho_1}(M_1+1)$$

hence, for $f\in C_{\rho_1}$ and for any $(t,u)\in\mathbb{R}^2$ with $\sqrt{t^2+u^2}\geq a_0$

(2.13) $st_A^2-\lim_{i,j}T_{ij}(f)=f(\rho_2-equal)$.

Also, using the Lemma 2.1, for any $a>0$ and for all $f\in C_{\rho_1}$, we have for any $(t,u)\in\mathbb{R}^2$ with $\sqrt{t^2+u^2}\leq a$,

(2.14) $st_A^2-\lim_{i,j}T_{ij}(f)=f(\rho_2-equal)$,

then, we immediately get from (2.13) and (2.14),

$$st_A^2-\lim_{i,j}T_{ij}(f)=f(\rho_2-equal)\text{ on }\mathbb{R}^2,$$

the proof is completed.

Using the similar consideration, we can get the following result;

Corollary 2.1. Let ρ_1 and ρ_2 be as in Lemma 2.1. Suppose that (T_{ij}) is a double sequence of positive linear operators from C_{ρ_1} into B_{ρ_2}. Then for all $f\in C_{\rho_1}$,

$P-\lim_{i,j}T_{ij}(f)=f(\rho_2-equal)$ on $\mathbb{R}^2$,

provided that

$P-\lim_{i,j}T_{ij}(F_r)=F_r(\rho_1-equal)$ on $\mathbb{R}^2$,

$r=0,1,2,3$.

Now, defining the weight function ρ_1 in Theorem 2.3 by $\rho_1(t,u)=1+t^2+u^2$ on $\mathbb{R}^2$, we study the rate of $A-$statistical equal convergence by using the following weighted modulus of continuity:

$$\omega_{\rho_1}(f,\delta)=\sup_{\sqrt{(s-t)^2+(v-u)^2}\leq\delta}\frac{|f(s,v)-f(t,u)|}{\rho_1(s,v)+\rho_1(t,u)},$$

where δ is a positive constant and $f\in C_{\rho_1}$ (see also [17, 7]). It can be easily seen that, for any $c>0$ and all $f\in C_{\rho_1}$

$$\omega_{\rho_1}(f,c\delta)\leq(1+[c])\,\omega_{\rho_1}(f,\delta)$$

where $[c]$ defined to be greatest integer less than or equal to c.

Following [17] and and if we use the same operators T_{ij} in Theorem 2.3, we can write, for any $\delta>0$, that

$$\begin{aligned}
&|T_{ij}(f;t,u)-f(t,u)|\leq T_{ij}(|f(s,v)-f(t,u)|;t,u)\\
&+|f(t,u)||T_{ij}(F_0;t,u)-F_0(t,u)|\\
&\leq T_{ij}\left(\rho_1(t,u)\,\omega_{\rho_1}\left(f,\delta\frac{\sqrt{(s-t)^2+(v-u)^2}}{\delta}\right);t,u\right)\\
&+|f(t,u)||T_{ij}(F_0;t,u)-F_0(t,u)|\\
&\leq\rho_1(t,u)\,\omega_{\rho_1}(f,\delta)\,T_{ij}\left(1+\frac{(s-t)^2+(v-u)^2}{\delta^2};t,u\right)\\
&+|f(t,u)||T_{ij}(F_0;t,u)-F_0(t,u)|\\
&\leq\rho_1(t,u)\,\omega_{\rho_1}(f,\delta)\left\{T_{ij}(\rho_1;t,u)+\frac{1}{\delta^2}T_{ij}\left(\varphi_{(t,u)};t,u\right)\right\}\\
&+|f(t,u)||T_{ij}(F_0;t,u)-F_0(t,u)|
\end{aligned}$$

where $\varphi_{(t,u)}(s,v):=(s-t)^2+(v-u)^2$ and we obtain that, for any $(t,u)\in\mathbb{R}^2$,

(2.15)

$$\begin{aligned}
&\left|\frac{T_{ij}(f;t,u)-f(t,u)}{\rho_2^2(t,u)}\right|\leq\|\rho_1\|_{\rho_2}\,\omega_{\rho_1}(f,\delta)\Bigg\{\|\rho_1\|_{\rho_2}\\
&+\frac{|T_{ij}(\rho_1;t,u)-\rho_1(t,u)|}{\rho_2(t,u)}+\frac{1}{\delta^2}\frac{T_{ij}\left(\varphi_{(t,u)};t,u\right)}{\rho_2(t,u)}\Bigg\}\\
&+\|f\|_{\rho_2}\|\rho_1\|_{\rho_2}\frac{|T_{ij}(F_0;t,u)-F_0(t,u)|}{\rho_1(t,u)}
\end{aligned}$$

provided that $T_{ij}\left(\varphi_{(t,u)}\right)\in B_{\rho_2}$.

Theorem 2.4. Let (T_{ij}) be the same as in Theorem 2.3. Let $T_{ij}\left(\varphi_{(t,u)}\right)\in B_{\rho_2}$ where

$\varphi_{(t,u)}(s,v):=(s-t)^2+(v-u)^2$. If

(i) $st_A^2-\lim_{i,j}T_{ij}(F_0)=F_0(\rho_1-equal)$ on $\mathbb{R}^2$,

(ii) $st_A^2-\lim_{i,j}\omega_{\rho_1}(f,\delta)=0(\rho_2-equal)$ on $\mathbb{R}^2$, where $\delta:=\sqrt{\frac{\left|T_{ij}\left(\varphi_{(t,u)};t,u\right)\right|}{\rho_2(t,u)}}$, then for all $f\in C_{\rho_1}$,

$$st_A^2-\lim_{i,j}T_{ij}(f)=f\left(\rho_2^2-equal\right)\text{ on }\mathbb{R}^2.$$

Proof. By (2.15), (i) and (ii), we get the desired result.

III. APPLICATION

We now present an example of a sequence of positive linear operators that satisfies the conditions of Theorem 2.3 but do not satisfy the conditions of Theorem 2.1, Theorem 2.2 and Corollary 2.1.

Example 3.1. Let us consider the following linear positive operators given in [18] which is defined by:

(3.1)

$$L_{ij}(f;t,u):=\sum_{v=0}^{\infty}\sum_{\mu=0}^{\infty}f\left(\frac{v}{\beta_i},\frac{\mu}{\gamma_j}\right)K_{i,v}(t)K_{j,\mu}(u)\frac{(-\alpha_i)^v}{v!}\frac{(-\alpha_j)^\mu}{\mu!}.$$

Here (α_i), (β_i) and (γ_i) be the real number sequences satisfying the followings:

(a) $\lim_{i\to\infty}\beta_i=\infty$ and $\lim_{j\to\infty}\gamma_j=\infty$,

(b) $\lim_{i\to\infty}\frac{\alpha_i}{\beta_i}=0$ and $\lim_{j\to\infty}\frac{\alpha_j}{\gamma_j}=0$,

(c) $\lim_{i\to\infty}i\frac{\alpha_i}{\beta_i}=1$ and $\lim_{j\to\infty}j\frac{\alpha_j}{\gamma_j}=1$,

and $K_{i,v}(t)$ and $K_{j,\mu}(u)$ are the functions satisfy the following conditions:

i) For any natural $i,j,v,\mu=0,1,2,...$ and for any $t,u\in[0,\infty)$

$$(-1)^v K_{i,v}(t)\geq0\text{ and }(-1)^\mu K_{j,\mu}(u)\geq0,$$

ii) For any $t,u\in[0,\infty)$

$$\sum_{v=0}^{\infty}K_{i,v}(t)\frac{(-\alpha_i)^v}{v!}=1\text{ and }\sum_{\mu=0}^{\infty}K_{j,\mu}(t)\frac{(-\alpha_j)^\mu}{\mu!}=1,$$

iii) $K_{i,v}(t)=-itK_{i+m,v-1}(t)$ and $K_{j,\mu}(u)=-juK_{j+n,\mu-1}(u)$ for any $t,u\in[0,\infty)$ where $i+m$, $j+n$ are natural numbers and m,n are constants independent of v,μ.

Then, using the operators $L_{ij}(f;t,u)$, introduce the following positive linear operators:

(3.2) $T_{ij}(f;t,u)=(1+g_{ij}(t,u))L_{ij}(f;t,u)$,

where (g_{ij}) given by (1.3) in Example 1.1. Also taking $A=C(1,1)$, $\rho_1(t,u)=1+t^2+u^2$ and $\rho_2(t,u)$ arbitrary such as satisfying the condition (1.2) holds. Then, we obtain the test functions $F_0(t,u)=1$, $F_1(t,u)=t$, $F_2(t,u)=u$ and $F_{23}(t,u)=t^2+u^2$. We claim that

(3.3) $st_A^2-\lim_{i,j} T_{ij}(F_r)=F_r(\rho_1-equal)$ on $\mathbb{R}^2$,

for each $r=0,1,2,3$.

Now observe that

$$T_{ij}(F_0;t,u)=\left(1+g_{ij}(t,u)\right)F_0(t,u),$$

$$T_{ij}(F_1;t,u)=\left(1+g_{ij}(t,u)\right)i\frac{\alpha_i}{\beta_i}F_1(t,u),$$

$$T_{ij}(F_2;t,u)=\left(1+g_{ij}(t,u)\right)j\frac{\alpha_j}{\gamma_j}F_2(t,u),$$

$$T_{ij}(F_3;t,u)=\left(1+g_{ij}(t,u)\right)\left[i(i+m)\frac{\alpha_i^2}{\beta_i^2}t^2+i\frac{\alpha_i}{\beta_i^2}t+j(j+n)\frac{\alpha_j^2}{\gamma_j^2}u^2+j\frac{\alpha_j}{\gamma_j^2}u\right].$$

Hence,

$$\left|T_{ij}(F_0;t,u)-F_0(t,u)\right|=\left|\left(1+g_{ij}(t,u)\right)-1\right|=g_{ij}(t,u),$$

and $st^2-\lim_{i,j} g_{ij}=0(\rho_1-equal)$ on $\mathbb{R}^2$, then,

$$st^2-\lim_{i,j} T_{ij}(F_0)=F_0(\rho_1-equal) \text{ on } \mathbb{R}^2$$

which guarantees that (3.3) holds true for $r=0$. It is obvious that

$$\left|T_{ij}(F_1;t,u)-F_1(t,u)\right|=\left|\left(1+g_{ij}(t,u)\right)i\frac{\alpha_i}{\beta_i}t-t\right|$$

$$=|t|\left|\left(1+g_{ij}(t,u)\right)i\frac{\alpha_i}{\beta_i}-1\right|$$

$$\le|t|\left(\left|i\frac{\alpha_i}{\beta_i}-1\right|+\left|g_{ij}(t,u)i\frac{\alpha_i}{\beta_i}\right|\right),$$

then, by virtue of (c) and $st^2-\lim_{i,j} g_{ij}=0(\rho_1-equal)$ on $\mathbb{R}^2$,

$$\lim_i\left(i\frac{\alpha_i}{\beta_i}-1\right)=0,$$

$$st^2-\lim_{i,j} g_{ij}i\frac{\alpha_i}{\beta_i}=0(\rho_1-equal) \text{ on } \mathbb{R}^2.$$

Since $\sup_{(t,u)\in[0,\infty)\times[0,\infty)}\frac{|t|}{1+t^2+u^2}<\infty$, we get

$$st^2-\lim_{i,j} T_{ij}(F_1)=F_1(\rho_1-equal) \text{ on } \mathbb{R}^2$$

which guarantees that (3.3) holds true for $r=1$. Similary we have

$$st^2-\lim_{i,j} T_{ij}(F_2)=F_2(\rho_1-equal) \text{ on } \mathbb{R}^2.$$

Finally, since

$$\left|T_{ij}(F_3;t,u)-F_3(t,u)\right|=\left|\left(1+g_{ij}(t,u)\right)\left[i(i+m)\frac{\alpha_i^2}{\beta_i^2}t^2+i\frac{\alpha_i}{\beta_i^2}t+j(j+n)\frac{\alpha_j^2}{\gamma_j^2}u^2+j\frac{\alpha_j}{\gamma_j^2}u\right]-t^2-u^2\right|$$

$$\le\left|t^2\right|\left(\left|i(i+m)\frac{\alpha_i^2}{\beta_i^2}-1\right|+\left|g_{ij}(t,u)i(i+m)\frac{\alpha_i^2}{\beta_i^2}\right|\right)$$

$$+\left|u^2\right|\left(\left|j(j+n)\frac{\alpha_j^2}{\gamma_j^2}-1\right|+g_{ij}(t,u)j(j+n)\frac{\alpha_j^2}{\gamma_j^2}\right)$$

$$+|t|\left|\left(1+g_{ij}(t,u)\right)i\frac{\alpha_i}{\beta_i^2}\right|+|u|\left|\left(1+g_{ij}(t,u)\right)j\frac{\alpha_j}{\gamma_j^2}\right|.$$

because of (c) and and $st^2-\lim_{i,j} g_{ij}=0(\rho_1-equal)$ on $\mathbb{R}^2$, we can easily see that

(3.4) $\lim_i\left(i(i+m)\frac{\alpha_i^2}{\beta_i^2}-1\right)=0,$

$$\lim_j\left(\left|j(j+n)\frac{\alpha_j^2}{\gamma_j^2}-1\right|\right)=0,$$

$$st^2-\lim_{i,j} g_{ij}i(i+m)\frac{\alpha_i^2}{\beta_i^2}=0(\rho_1-equal) \text{ on } \mathbb{R}^2,$$

$$st^2-\lim_{i,j} g_{ij}j(j+n)\frac{\alpha_j^2}{\gamma_j^2}=0(\rho_1-equal) \text{ on } \mathbb{R}^2,$$

and since $\lim_i\frac{1}{\beta_i}=0$, $\lim_j\frac{1}{\gamma_j}=0$ from (a) and also using (c), we get

(3.5) $\lim_i i\frac{\alpha_i}{\beta_i^2}=0,\ \lim_j j\frac{\alpha_j}{\gamma_j^2}=0,$

$$st^2-\lim_{i,j} g_{ij}i\frac{\alpha_i}{\beta_i^2}=0(\rho_1-equal) \text{ on } \mathbb{R}^2,$$

$$st^2-\lim_{i,j} g_{ij}j\frac{\alpha_j}{\gamma_j^2}=0(\rho_1-equal) \text{ on } \mathbb{R}^2.$$

Using (3.4) and (3.5) and since $\sup_{(t,u)\in[0,\infty)\times[0,\infty)}\frac{|t|^2}{1+t^2+u^2}<\infty$, $\sup_{(t,u)\in[0,\infty)\times[0,\infty)}\frac{|u|^2}{1+t^2+u^2}<\infty$, $\sup_{(t,u)\in[0,\infty)\times[0,\infty)}\frac{|t|}{1+t^2+u^2}<\infty$ and $\sup_{(t,u)\in[0,\infty)\times[0,\infty)}\frac{|u|}{1+t^2+u^2}<\infty$, we can write

$$st^2-\lim_{i,j} T_{ij}(F_3)=F_3(\rho_1-equal) \text{ on } \mathbb{R}^2.$$

Hence, our claim (3.3) holds true for each $r=0,1,2,3$. (T_{ij}) satisfies all hypothesis of Theorem 2.3 and we immediately see that, for all $f\in C_{\rho_1}$,

$$st^2-\lim_{i,j}T_{ij}\left(f\right)=f\left(\rho_2-equal\right) \text{ on } \mathbb{R}^2.$$

However, since

$$\frac{\left|T_{ij}\left(F_0;t,u\right)-F_0\left(t,u\right)\right|}{\rho_1\left(t,u\right)}$$

$$=\frac{\left|\left(1+g_{ij}\left(t,u\right)\right)F_0\left(t,u\right)-F_0\left(t,u\right)\right|}{\rho_1\left(t,u\right)}=\frac{g_{ij}\left(t,u\right)}{\rho_1\left(t,u\right)},$$

then Theorem 2.1, Theorem 2.2 and Corollary 2.1 do not work for the sequence $\left(T_{ij}\right)$.

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