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Equi-Statistical Relative Convergence and Korovkin-Type Approximation

Sevda Yıldız[1*+], Kamil Demirci[1] and Fadime Dirik[1]

[1]*Department of Mathematics, Sinop University, Sinop, Turkey*
[*]*Corresponding author:* sevdaorhan@sinop.edu.tr
[+]*Speaker:* sevdaorhan@sinop.edu.tr
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Abstract – Classical approximation theory has started with the proof of Weierstrass approximation theorem and after that Korovkin [Linear operators and approximation theory, Hindustan Publ. Corp, Delhi, 1960] first established the necessary and sufficient conditions for uniform convergence of a sequence of positive linear operators to a function f. In classical Korovkin theorem, most of the classical operators tend to converge to the value of the function being approximated. Also, the attention of researchers has been attracted to the notion of statistical convergence because of the fact that it is stronger than the classical convergence method. Furthermore, the concept of equi-statistical convergence is more general than the statistical uniform convergence. In this work, we introduce our new convergence method named equi-statistical relative convergence to demonstrate a Korovkin type approximation theorems which were proven by earlier authors. Then, we present an example in support of our definition and result presented in this paper. Finally, we compute the rate of the convergence.

Keywords – Korovkin Theorem, Modular Spaces, Statistical Equal Convergence

I. Introduction and Preliminaries

Attention of researchers has been attracted to statistical convergence ([1, 2]) because of the fact that it is stronger than the classical convergence. Furthermore, the concept of equi-statistical convergence ([3]) is more general than statistical uniform convergence. Recently, Demirci and Orhan ([4]) define a new type of statistical convergence by using the notions of the natural density ([5]) and the relative uniform convergence ([6, 7]). The main purpose of the present paper is using these toughts for defining equi-statistical relative convergence and using this convergence method to prove a Korovkin type theorem and also, giving equi-statistical relative rates.

Now, we observe that the space $C(X)$ of all continuous real-valued functions defined on a compact subset X of real numbers is also a Banach space. For $f \in C(X)$, we have

$$\|f\|_{C(X)} := \sup_{t\in X} |f(t)| .$$

Let f and f_r (for $\forall r \in \mathbb{N}$) belong to $C(X)$.

Definition 1.1. ([8]) (f_r) is said to be pointwise statistically convergent to f on X if for every $\varepsilon > 0$ and for each $t \in X$,

$$\lim_{r\to\infty} \frac{\Omega_r(t,\varepsilon)}{r} = 0$$

where $\Omega_r(t,\varepsilon) := \left|\left\{k \le r : \left|f_k(t) - f(t)\right| \ge \varepsilon\right\}\right|$. In this case we write $f_r \to f(st)$ on X.

Definition 1.2. ([8]) (f_r) is said to be uniform statistically convergent to f on X if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{\phi_r(\varepsilon)}{r} = 0,$$

where $\phi_r(\varepsilon) := \left|\left\{k \le r : \|f_k - f\|_{C(X)} \ge \varepsilon\right\}\right|$. In this case we write $f_r \rightrightarrows f(st)$ on X.

Now, we remind the concept of equi-statistical convergence given by Balcerzak et al. [3]:

Definition 1.3. ([3]) (f_r) is said to be equi-statistically convergent to f on X if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{\Omega_r(t,\varepsilon)}{r} = 0,$$ uniformly with respect to $t \in X$. In this case we write $f_r \to f(equi-st)$ on X.

Using the above definitions, the following result given by Balcerzak et al. [3];

Lemma 1.1. ([3]) $f_r \rightrightarrows f$ on X implies $f_r \rightrightarrows f(st)$ on X, which also implies $f_r \to f(equi-st)$ on X.

Now we give the Demirci and Orhan's definition mentioned above.

Definition 1.4. ([4]) (f_r) is said to be statistically relatively uniform convergent to f on X if there exists a function $\sigma(t)$, called skale function, $|\sigma(t)| > 0$, such that, for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{\Delta_r(\varepsilon)}{r} = 0,$$

where $\Delta_r(\varepsilon) := \left|\left\{k \le r : \left\|\frac{f_k - f}{\sigma}\right\|_{C(X)} \ge \varepsilon\right\}\right|$. This limit denoted by $f_r \rightrightarrows f(st)(X;\sigma)$. This limit is denoted by $f_r \rightrightarrows f\ (st)(X;\sigma)$.

Now, we introduce the concept of the equi-statistical relative convergence of sequences of functions with the help of the Definition 1.3 and Definition 1.4.

Definition 1.5. (f_r) is said to be equi-statistically relatively convergent to f on X if there exists a function $\sigma(t)$, called skale function, $|\sigma(t)|>0$, such that, for every $\varepsilon>0$,

$$\lim_{r\to\infty}\frac{\psi_r(t,\varepsilon)}{r}=0, \text{ uniformly with respect to } t\in X,$$

where $\psi_r(t,\varepsilon):=\left|\left\{k\le r:\left|\frac{f_k(t)-f(t)}{\sigma(t)}\right|\ge\varepsilon\right\}\right|$. In this case we write $f_r\to f(equi-st)(X;\sigma)$.

Remark 1.1. It will be observed that equi-statistical convergence is the special case of equi-statistical relative convergence in which the scale function is a non-zero constant.

Now, we give the following example which is an equi-statistically relatively convergent but not statistically uniform (or uniform) convergent.

Example 1.1. Let $X=[0,1]$ and g is a function by $g(t)=0$ for $t\in[0,1]$. For each $r\in\mathbb{N}$, define $g_r\in C[0,1]$ by

$$g_r(t)=\begin{cases}2^{r+1}t-2, & \frac{1}{2^r}\le t\le\frac{3}{2^{r+1}},\\ -2^{r+1}t+4, & \frac{3}{2^{r+1}}\le t\le\frac{1}{2^{r-1}},\\ 0, & otherwise.\end{cases} \tag{1.1}$$

Take σ defined by $\sigma(t):=\begin{cases}1, & t=0,\\ \frac{1}{t}, & 0<t\le1.\end{cases}$ Thus, clearly,

$$\frac{g_r(t)-g(t)}{\sigma(t)}=\begin{cases}2^{r+1}t^2-2t, & \frac{1}{2^r}\le t\le\frac{3}{2^{r+1}},\\ -2^{r+1}t^2+4t, & \frac{3}{2^{r+1}}\le t\le\frac{1}{2^{r-1}},\\ 0, & otherwise.\end{cases}$$

Then observe that $g_r\to g=0(equi-st)([0,1];\sigma)$, however, since $\sup_{t\in[0,1]}|g_r(t)-g(t)|=1$, (g_r) is not statistically (or ordinary) uniform convergent to $g=0$.

II. Korovkin Theorem via Equi-Statistical Relative Convergence

Korovkin-type approximation has been widely studied in the literature ([9]). The Korovkin-type approximation theorem for sequences of positive linear operators defined on $C[a,b]$ has been proved via the concept of statistical convergence in [10]. In this section, we give a Korovkin-type theorem for sequences of positive linear operators defined on $C(X)$ using the concept of equi-statistical relative convergence.

Let L be a linear operator from $C(X)$ into itself. Then, as usual, we say that L is positive linear operator provided that $f\ge0$ implies $L(f)\ge0$. Also, we denote the value of $L(f)$ at a point $t\in X$ by $L(f(u);t)$ or, briefly $L(f;t)$.

Theorem 2.1. Let (L_r) be a sequence of positive linear operators acting $C(X)$ into $C(X)$. Then, we have

$$L_r(f)\to f(equi-st)(X;\sigma) \tag{2.1}$$

if and only if

$$L_r(e_i)\to e_i(equi-st)(X;\sigma_i),\ i=0,1,2, \tag{2.2}$$

where $e_i(t)=t^i$, $i=0,1,2$, $|\sigma_i(t)|>0: i=0,1,2$, and $\sigma(t):=\max\{|\sigma_i(t)|: i=0,1,2\}$.

Proof. Since each of the functions given by

$e_i(t)=t^i$, $i=0,1,2,$

belong to $C(X)$, the implication (2.1) $\Rightarrow$ (2.2) are quite obvious. Before to complete the proof of Theorem 2.1, we assume that (2.2) are hold. Let f belong to $C(X)$ and $t\in X$ fixed. Then there exists a constant $\kappa>0$ such that for every $t\in X$,

$$|f(t)|\le\kappa$$

which ensure that

$$|f(u)-f(t)|\le2\kappa.$$

From the continuity of f, for a given $\varepsilon>0$, there exists $\delta>0$ such that

$$|f(u)-f(t)|<\varepsilon \text{ whenever } |u-t|<\delta \tag{2.3}$$

for every $u,t\in X$. Now let us select

$$\phi(u,t)=(u-t)^2.$$

If $|u-t|\ge\delta$, $u,t\in X$, we get

$$|f(u)-f(t)|<\frac{2\kappa}{\delta^2}\phi(u,t). \tag{2.4}$$

From (2.3) and (2.4), we obtain

$$|f(u)-f(t)|<\varepsilon+\frac{2\kappa}{\delta^2}\phi(u,t)$$

i.e.

$$-\varepsilon-\frac{2\kappa}{\delta^2}\phi(u,t)<f(u)-f(t)<\varepsilon+\frac{2\kappa}{\delta^2}\phi(u,t). \tag{2.5}$$

Since the positive linear operator $L_r(1;t)$ is monotone, by applying this operator to the inequality in (2.5), we have

(2.6)
$$L_r(1;t)\left(-\varepsilon-\frac{2\kappa}{\delta^2}\phi(u,t)\right)<L_r(1;t)\left(f(u)-f(t)\right)<L_r(1;t)\left(\varepsilon+\frac{2\kappa}{\delta^2}\phi(u,t)\right).$$

Then, we obtain

(2.7)
$$-\varepsilon L_r(1;t)-\frac{2\kappa}{\delta^2}L_r(\phi(u,t);t)<L_r(f;t)-f(t)L_r(1;t)<\varepsilon L_r(1;t)+\frac{2\kappa}{\delta^2}L_r(\phi(u,t);t).$$

Since

$$L_r(f;t)-f(t)=\left[L_r(f;t)-f(t)L_r(1;t)\right]+f(t)\left[L_r(1;t)-1\right], \tag{2.8}$$

we apply the equality (2.8) in (2.7) ,

(2.9)
$$L_r(f;t)-f(t)\le\varepsilon L_r(1;t)+\frac{2\kappa}{\delta^2}L_r(\phi(u,t);t)+f(t)\left[L_r(1;t)-1\right].$$

Now we calculate the term of "$L_r(\phi(u,t);t)$" then we write

(2.10)
$$\begin{aligned}L_r(\phi(u,t);t)&=L_r\left((u-t)^2;t\right)\\&=L_r\left(u^2-2tu+t^2;t\right)\\&=L_r\left(u^2;t\right)-2tL_r(u;t)+t^2L_r(1;t)\\&=\left[L_r\left(u^2;t\right)-t^2\right]-2t\left[L_r(u;t)-t\right]\\&\quad+t^2\left[L_r(1;t)-1\right].\end{aligned}$$

By using (2.10), we write the following inequality,

$$L_r(f;t)-f(t)\le\varepsilon+(\varepsilon+f(t))\left[L_r(1;t)-1\right]+\frac{2\kappa}{\delta^2}\left\{\left[L_r\left(u^2;t\right)-t^2\right]-2t\left[L_r(u;t)-t\right]+t^2\left[L_r(1;t)-1\right]\right\}.$$

Since $\varepsilon>0$ is arbitrary, we can write

$$\left|L_r(f;t)-f(t)\right|\le F\left\{\left|L_r(1;t)-1\right|+\left|L_r(u;t)-t\right|+\left|L_r\left(u^2;t\right)-t^2\right|\right\}$$

where $F:=\max\left\{\varepsilon+\kappa+\frac{2\kappa}{\delta^2}\|e_2\|,\frac{4\kappa}{\delta^2}\|e_1\|,\frac{2\kappa}{\delta^2}\right\}$. Let $\sigma(t):=\max\left\{|\sigma_i(t)|:i=0,1,2\right\}$ and $|\sigma_i(t)|>0:i=0,1,2$. Hence we get

$$\left|\frac{L_r(f;t)-f(t)}{\sigma(t)}\right|\le F\left\{\left|\frac{L_r(e_0;t)-e_0(t)}{\sigma_0(t)}\right|+\left|\frac{L_r(e_1;t)-e_1(t)}{\sigma_1(t)}\right|+\left|\frac{L_r(e_2;t)-e_2(t)}{\sigma_2(t)}\right|\right\}$$

Now, for a given $\varepsilon'>0$, define the following sets:

$$\psi_r(t,\varepsilon'):=\left|\left\{k\le r:\left|\frac{L_k(f;t)-f(t)}{\sigma(t)}\right|\ge\varepsilon'\right\}\right|$$

and

(2.11)
$$\psi_{i,r}\left(t,\frac{\varepsilon'}{3F}\right):=\left|\left\{k\le r:\left|\frac{L_k(e_i;t)-e_i(t)}{\sigma_i(t)}\right|\ge\frac{\varepsilon'}{3F}\right\}\right|,$$

($i=0,1,2$). It is easy to see that

$\frac{\psi_r(t,\varepsilon')}{r}\le\sum_{i=0}^{2}\frac{\psi_{i,r}\left(t,\frac{\varepsilon'}{3F}\right)}{r}$. Then using the hypothesis (2.2) and considering Definition 1.5, the right hand side of (2.11) tend to zero as $r\to\infty$. The proof is completed.

III. APPLICATION

Let $X=[0,1]$. Consider the following the Meyer-König and Zeller polynomials introduced by W. Meyer-König and K. Zeller [11]:

$$M_r(f;t)=\sum_{k=0}^{\infty}f\left(\frac{k}{r+k}\right)\binom{r+k}{k}t^k(1-t)^{r+1},$$

$f\in C[0,1]$. It is known that

$M_r(e_i;t)=e_i(t)$, $i=0,1$

$$M_r(e_2;t)=e_2(t)+\eta_r(t)\le e_2(t)+\frac{t(1-t)}{r+1},$$

where $\eta_r(t)=t(1-t)^{r+1}\sum_{k=1}^{\infty}\frac{(r+k-1)!}{(r-1)!k!}\frac{t^k}{r+k+1}$.

Using the polynomial, $D_r:C[0,1]\to C[0,1]$ be a sequence of operators defined as follows:

(3.1) $D_r(f;t)=(1+g_r(t))M_r(f;t)$, $t\in[0,1]$ and $f\in C[0,1]$, where $g_r(t)$ given by (1.1) in Example 1.1. Then, we see that

$$D_r(e_0;t)=(1+g_r(t))e_0(t),$$
$$D_r(e_1;t)=(1+g_r(t))e_1(t),$$
$$D_r(e_2;t)\le(1+g_r(t))\left[e_2(t)+\frac{t(1-t)}{r+1}\right].$$

Since

$g_r\to g=0(equi-st)([0,1];\sigma)$,

we conclude that

$D_r(e_i)\to e_i(equi-st)([0,1];\sigma)$ for each $i=0,1,2$.

So, by Theorem 2.1, we have

$D_r(f) \to f(equi-st)([0,1];\sigma)$.

Furthermore, since the sequence (g_r) of functions on $[0,1]$ is not statistically (or ordinary) uniform convergent to the function $g=0$ on the interval $[0,1]$; we can say that the results given in [10] and [9], respectively, do not hold true for our operators defined by (3.1).

IV. Rate of the equi-statistical relative convergence

In this section, we compute the rate of equi-statistical relative convergence with the help of modulus of continuity. Now, we recall that the modulus of continuity of a function $f \in C(X)$ is denoted by $\omega(f;\delta)$, is defined to be

$$\omega(f;\delta) = \sup_{|u-t|\le\delta,\ t,u\in X} |f(u)-f(t)| \quad (\delta>0).$$

It is also well known that for any $\delta>0$ and each $u,t \in X$

$$|f(u)-f(t)| \le \omega(f;\delta)\left(\frac{|u-t|}{\delta}+1\right).$$

Now, we state and prove the following theorem.

Theorem 4.1. Let (L_r) be a sequence of positive linear operators acting $C(X)$ into $C(X)$. Assume that the following conditions hold:

(i) $L_r(e_0) \to e_0(equi-st)(X;\sigma_0)$,

(ii) $\omega(f;\alpha_r) \to 0(equi-st)(X;\sigma_1)$, where $\alpha_r(t)=\sqrt{L_r(\varphi_t;t)}$ with $\varphi_t(u)=(u-t)^2$.

Then we have, for all $f \in C(X)$,

(4.1) $\qquad L_r(f) \to f(equi-st)(X;\sigma)$

where $|\sigma_i(t)|>0 : i=0,1$ and

$$\sigma(t) := \max\{|\sigma_0(t)|, |\sigma_1(t)|, |\sigma_0(t)\sigma_1(t)|\}.$$

Proof. Let $f \in C(X)$ and $t \in X$. It is known that ([12], [13]),

$$|L_r(f;t)-f(t)| \le \kappa |L_r(e_0;t)-e_0(t)| + \left\{L_r(e_0;t)+\sqrt{L_r(e_0;t)}\right\}\omega(f;\alpha_r)$$

where $\kappa := \|f\|_{C(X)}$. Then, we obtain

$$|L_r(f;t)-f(t)| \le \kappa|L_r(e_0;t)-e_0(t)| + 2\omega(f;\alpha_r) + |L_r(e_0;t)-e_0(t)|\omega(f;\alpha_r) + \sqrt{|L_r(e_0;t)-e_0(t)|}\,\omega(f;\alpha_r).$$

This yields that

$$\left|\frac{L_r(f;t)-f(t)}{\sigma(t)}\right| \le \kappa\left|\frac{L_r(e_0;t)-e_0(t)}{\sigma_0(t)}\right| + 2\frac{\omega(f;\alpha_r)}{|\sigma_1(t)|} + \left|\frac{L_r(e_0;t)-e_0(t)}{\sigma_0(t)}\right|\frac{\omega(f;\alpha_r)}{|\sigma_1(t)|} + \sqrt{\left|\frac{L_r(e_0;t)-e_0(t)}{\sigma_0(t)}\right|}\,\frac{\omega(f;\alpha_r)}{|\sigma_1(t)|}.$$

Now, considering the above inequality, the hypotheses (i) and (ii), proof is completed at once.

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