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Coercive Solvability of Many-Interval Sturm-Liouville Problems

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Abstract. It is well-known that the Sturm-Liouville type boundary value problems appears in solving many important problems in physics. For example, some problems of theory of elasticity in a half-strip ,the problems of theory of vibrations of an elastic cylinder reduce to investigation of solvability of appropriate boundary value problems for Sturm-Liouville type differential equations. In the recent years, the Sturm-Liouville type boundary value problems with additional transmission conditions are investigated by many mathematical and physical researches . Note that, such type problems is very complicated because boundary value problems with additional transmission conditions may be not self-adjoint in the classical Hilbert space L_2 and therefore the eigenvalues may be not real.

The main goal of this study is to prove coerciveness of a new class many interval Sturm-Liouville problems with additional transmission conditions at the points of interaction. Moreover we shall establish some spectral properties and find asymptotic behaviour of the eigenvalues of the problem under consideration.

Observe that this type of problems is studied in the setting of the direct sum of the Hilbert space. We also construct fundamental solutions and discuss some properties of spectrum.

Keywords : Boundary value problem, transmission conditions, Coercive solvability, spectrum, Hilbert space.

Introduction

It is well-known that the Sturm-Liouville type boundary value problems appears in solving many important problems in physics. For example, some problems of theory of elasticity in a half-strip [10, 11, 15] the problems of theory of vibrations of an elastic cylinder [1] reduce to investigation of solvability of appropriate boundary value problems for Sturm-Liouville type differential equations. We also refer to [1, 5] for various physical applications.

In the recent years, the Sturm-Liouville type boundary value problems with additional transmission conditions are concerned by many mathematical and physical researches (see, for example [2, 3, 6, 8, 12, 13, 16] and references, cited therein). Note that, such type problems is very complicated because boundary value problems with additional transmission conditions may be not self-adjoint in the classical Hilbert space L_2 and therefore the eigenvalues may be not real. In this study we shall investigate the coercive solvability of a new class the Sturm-Liouville equations on disjoint intervals with additional transmission conditions at the points of interaction. Observe that this type of problems is studied in the setting of the direct sum of the Hilbert space. We also costruct fundamental solutions and discuss some properties of spectrum.

Statement of the problem and some auxiliary definitions and results

In this study we shall investigate some spectral aspects of the differential-operator equation

$$L_0u(x,\lambda) + Au(x,\lambda) = \lambda\rho(x)u(x,\lambda) \tag{1}$$

on the many-interval $\Omega = \cup_{i=1}^{3}\Omega_i$ where $\Omega_1 = (a,c_1),\ \Omega_2 = (c_1,c_2),\ \Omega_3 = (c_2,b),\ a < c_1 < c_2 < b$, subject to the boundary conditions given by

$$\ell_1 u := \alpha_1 u(a,\lambda) + \beta_1 \frac{\partial u(a,\lambda)}{\partial x} = 0, \tag{2}$$

$$\ell_2 u := \alpha_2 u(b,\lambda) + \beta_2 \frac{\partial u(b,\lambda)}{\partial x} = 0, \tag{3}$$

and additional transmission conditions given by

$$t_{i1} := \delta_{i1} \frac{\partial u(c_i - 0,\lambda)}{\partial x} + \gamma_{i1} u(c_i + 0,\lambda) = 0, \tag{4}$$

$$t_{i2} := \delta_{i2} u(c_i - 0,\lambda) + \gamma_{i2} \frac{\partial u(c_i + 0,\lambda)}{\partial x} = 0, \tag{5}$$

where $L_0u := \frac{\partial^2 u(x,\lambda)}{\partial x^2}$, A is abstract linear operator in the direct sum of Hilbert spaces $\oplus_{i=1}^{3} L_2(\Omega_i);\ \rho(x) = \rho_i^2$ for $x \in \Omega_i$; $\rho_i, \alpha_i, \beta_i, \gamma_{ij}$ and δ_{ij} nonzero are real numbers; λ is a complex spectral parameter.

The problem (1)-(5) is a non-classical Sturm-Liouville problem since it contains abstract linear operator A (the operator A may be non-differential) and additional transmission conditions (4)-(5). Note that such type of boundary-value problems arise after an application of the method of separation of variables to the varied assortment of physical problems. The development of classical theory of linear differential operators can be found in the books of Naimark [9], Coddington and Levinson [4] and Levitan and Sargsyan [7].

Preliminaries

Let E_1 and E_2 be two Banach spaces both linearly and continuously embedded in a Banach space E. The pair $\{E_1,E_2\}$ is said to be an interpolation couple. It can be shown easily that the linear space $E_1+E_2 := \{u = u_1+u_2 \mid u_1 \in E_1, u_2 \in E_2\}$ equipped with the norm $\|u\|_{E_1+E_2} := \inf\{\|u_1\|_{E_1} + \|u_2\|_{E_2} \mid u_1 \in E_1, u_2 \in E_2, u = u_1+u_2\}$ is the Banach space. Defining $K(t,u) = \inf\{\|u_1\|_{E_1} + \|u_2\|_{E_2} \mid u_1 \in E_1, u_2 \in E_2, u = u_1+u_2\}$ for $u \in E_1 + E_2$ and $t > 0$, we see that this functional is continuous with respect to the parameter t.

Definition 1 *Let $\{E_1,E_2\}$ be any interpolation couple and θ and p are any real numbers with $0<\theta<1,\ 1 \le p < \infty$. Then the space $\{E_1,E_2\}_{\theta,p}$ defined by*

$$(E_1,E_2)_{\theta,p} := \left\{ u = u_1+u_2 \mid u_1 \in E_1, u_2 \in E_2,\ \|u\|_{(E_1,E_2)_{\theta,p}} := \left(\int_0^\infty \frac{K^p(t,u)}{t^{1+\theta p}} dt \right)^{\frac{1}{p}} < \infty \right\}$$

is said to be $(\theta,p)-$ interpolation space for the interpolation couple $\{E_1,E_2\}$ by the $K-$ method (see, for example [14]).

Lemma 1 *There exists a constant $C_{\theta,p} > 0$ such that the inequality*

$$\|u\|_{\theta,p} \le C_{\theta,p} \|u\|_{E_1}^{1-\theta} \|u\|_{E_2}^{\theta} \tag{6}$$

holds for all $u \in E_1 \cap E_2$ (see, [14]), where by $\|u\|_{\theta,p}$ we mean the norm of $(E_1,E_2)_{\theta,p}$.

Let $W_2^k(a,b), k = 1,2,...,$ be well-known Sobolev spaces, $s > 0$ is non-integer real number and n be any integer with $n > s$. Then the space $W_2^s(a,b)$ is defined as interpolation space

$$W_2^s(a,b) := \left(W_2^n(a,b), L_2(a,b)\right)_{1-\frac{s}{n},2}. \tag{7}$$

Remark 1 *It is easy to see that, the equality (7) holds even in the case when s is also integer number.*

Below by $\|u\|_{\Omega,0}, \|u\|_{\Omega,2}, \|u\|_{\Omega,s}$ *we shall denote the norms of the spaces* $\oplus_{i=1}^3 L_2(\Omega_i)$, $\oplus_{i=1}^3 W_2^2(\Omega_i)$ *and* $\oplus_{i=1}^3 W_2^s(\Omega_i)$ *respectively, where* $0 < s < 2$.

Theorem 1 *For each* $0 < s < 2$ *there is a constant* $m_s > 0$ *such that*

$$\|u\|_{\Omega,s} \;\leq\; m_s\Big(|\lambda|^s\|u\|_{\Omega,0} + \frac{1}{|\lambda|^{2-s}}\|u\|_{\Omega,s}\Big).$$

Let us recall some definition and facts.

Definition 2 *[17] Let* $0 \leq p \leq 1$. *The operator B is said to be p-subordinate to T if* $D(B) \supset D(A)$ *and there is* $c > 0$ *such that*

$$\|Bu\| \;\leq\; c\,\|Tu\|^p\,\|u\|^{1-p}$$

for all $u \in D(A)$.

Definition 3 *[17] The operator B is called A− compact if* $D(B) \supset D(A)$ *and the operator* $B(A-\lambda_0 I)^{-1}$ *is compact for some regular point* $\lambda = \lambda_0$ *of A.*

Theorem 2 *[17] Suppose that the operator T is self-adjoint operator with discrete spectrum and B is p-subordinate to T for some* $0 < p < 1$. *Then B is T− compact.*

Theorem 3 *[17] Let T be self-adjoint operator with discrete spectrum and let B is T− compact. Then the spectrum of the operator* $T + B$ *is discrete.*

Main results

By using the above results we can establish the following main results. Consider the following nonhomogeneous problem

$$L_0u(x) + Au(x) + \lambda\rho(x)u(x) = f(x), \tag{8}$$

$$R_iu = f_i \;,\quad i = 1,2, \tag{9}$$

$$t_{ij}u = g_{ij} \;,\quad i,j = 1,2. \tag{10}$$

Everywhere in below we shall assume that

$$\frac{\gamma_{i1}\,\gamma_{i2}}{\rho_i^2} \;=\; \frac{\delta_{i1}\,\delta_{i2}}{\rho_{i+1}^2} \quad,\qquad i = 1,2.$$

Theorem 4 *Let the operator A from* $\oplus_{i=1}^3 W_2^2(\Omega_i)$ *to* $\oplus_{i=1}^3 L_2(\Omega_i)$ *is compact. Then for any* $\varepsilon > 0$ *there exists* $R_\varepsilon > 0$ *such that for all complex numbers* λ *satisfying* $|\arg\lambda \pm \frac{\pi}{2}| \;>\; \varepsilon,\ |\lambda| \;>\; R_\varepsilon$, *the operator*

$\Upsilon(\lambda) := u \;\rightarrow\; \Big(L_0u + Au + \lambda\rho u\;,\; \ell_1 u\;,\; \ell_2 u\;,\; t_{11}\;,\; t_{12}\;,\; t_{21}\;,\; t_{22}\Big)$ *from* $\oplus_{i=1}^3 W_2^2(\Omega_i)$ *onto* $\oplus_{i=1}^3 L_2(\Omega_i) \oplus \mathbb{C}^6$ *is an isomorphism and for these* λ *the following coercive estimate holds for the solution of the problem (8)-(10):*

$$|\lambda|\|u\|_{\Omega,0} \;+\; |\lambda|^{\frac{1}{2}}\|u\|_{\Omega,1} \;+\; \|u\|_{\Omega,2} \;\leq\; c(\varepsilon)\left(\|f\|_{\Omega,0} \;+\; |\lambda|^{\frac{1}{4}}\left(\sum_{i=1}^{2}|f_i| + \sum_{i,j=1}^{2}|g_{i,j}|\right)\right)$$

where $c(\varepsilon) > 0$ *is constant depending only on* $\varepsilon > 0$.

Now define the linear operator $\widetilde{L}_0$ in the direct sum space $\oplus_{i=1}^{3} L_2(\Omega_i)$ by action low $\widetilde{L}_0 u = u''$ on the domain

$$D(\widetilde{L}_0) = \left\{ u \mid u, u' \in \oplus_{i=1}^{3} AC_{loc}(\Omega_i), L_0 u \in \oplus_{i=1}^{3} L_2(\Omega_i), \ell_i u = 0 \ (i = 1, 2), t_{ij} u = 0 \ (i, j = 1, 2) \right\}.$$

Theorem 5 *The operator $\widetilde{L}_0$ is densely defined, i.e.* $\overline{D(\widetilde{L}_0)} = \oplus_{i=1}^{3} L_2(\Omega_i)$.

Theorem 6 *The operator $\widetilde{L}_0$ is self-adjoint.*

Corollary 1 *All eigenvalues of $\widetilde{L}_0$ are real.*

Theorem 7 *Suppose that the operator A is $\widetilde{L}_0-$ compact. Then the spectrum of $\widetilde{L} := \widetilde{L}_0 + A$ is discrete and consist of precisely denumerable many eigenvalues.*

Theorem 8 *Suppose that there is a constant $c > 0$ such that*

$$||Au||_{\Omega,0} \leq c \, ||u||_{\Omega,1}.$$

Then the boundary-value transmission problem (1)-(5) has precisely denumerable many eigenvalues $\lambda_1, \lambda_2, ...$ with the following asymptotic behaviour

$$\lambda_n = \omega \pi^2 n^2 + O(n)$$

for some real number ω.

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