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Hilbert Space Method for Sturm-Liouville Problems with Discontinuities

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Abstract. Boundary-value problems with eigenparameter dependent boundary conditions is one of the most extensively developing fields in pure and applied mathematics. First, we cite the works of Walter and Fulton, both of which have extensively bibliographies, a discussion of physical applications. Walter had given an Hilbert space formulation of such type eigenvalue problems and obtained the expansion result using the selfadjointness of the operator associated with the Sturm-Liouville problem. In recent years there has been increasing interest of Sturm-Liouville type problems with supplementary transmission conditions. For example, the first author of this study and some others have developed the spectral theory of discontinuous Sturm-Liouville problems with additional transmission conditions at the interior points. The concept of generalized solutions in a Hilbert space allows the eigenvalue problem to be reduced to an operator-pencil equation. In this study, we prove that the weak eigenfunctions of the Sturm-Liouville problem with additional transmission conditions form a Riesz basis of the corresponding Hilbert space. First, the generalized solution of the Sturm-Liouville problem is defined as an element of a direct sum spaces with satisfies some integral equalities. Second, using the Riesz representation theorem these equalities are reduced to an operator-pencil equation. Finally, it is established that the eigenfunctions of the original boundary-value-transmission problem form a Riesz basis of suitable Hilbert space.

Keywords : Sturm-Liouville problems, transmission conditions, eigenvalue, discontinuity, Hilbert space.

Introduction

Boundary-value problems with eigenparameter dependent boundary conditions is one of the most extensively developing fields in pure and applied mathematics. First, we cite the works of Walter[21] and Fulton[6], both of which have extensively bibliographies, in the case of [6], a discussion of physical applications. Walter[21] had given an Hilbert space formulation of such type eigenvalue problems and obtained the expansion result using the selfadjointness of the operator associated with the Sturm-Liouville problem. Fulton[6] established that the analysis of Titchmarsh's method[20] for regular Sturm-Liouville problem carries over to similar problems involving the eigenvalue parameter linearly in the boundary conditions.

The general results about operator polynomials including the multidimensional case can be found in monographs by Gohberg and Krein [7] and Rodman [19]. Greenberg and Babuska [8] considered Sturm-Liouville problems for second-order and fourth-order differential equations. Here the eigenvalue may occur nonlinearly in the differential operator and in the boundary conditions. Using variational properties of eigenvalues and the Sturm comparison theorems, the authors found bounds for eigenvalues.

In recent years there has been increasing interest of such type problems but under supplementary transmission conditions at the some interior points. For example, the first author of this paper and some others have developed the spectral theory of boundary-value-transmission problems (see, [1, 2, 3, 9, 13, 14, 15, 16, 17, 18, 22]). Such properties as isomorphism, coerciveness with respect to the spectral parameter, completeness and Abel basis property of a system of root functions, asymptotics of eigenvalues of some boundary value problems with transmission conditions and its applications to the corresponding initial boundary value problems for parabolic equations have been investigated in [13, 14, 15, 16].

The concept of generalized (weak) solutions in a Hilbert space allows the eigenvalue problem to be reduced to an operator-pencil equation (see, [12]). Keldysh[10] considered a wide class of operator equations containing a spectral parameter of the operator-pencil form

$$\mathrm{T}(\lambda)\psi = 0, \quad \mathrm{T}(\lambda) := \sum_{i=0}^{n} \lambda^i A_i \tag{1}$$

in a suitable Hilbert space. Belinskiy and Dauer [4, 5] have considered the eigenfunctions of a regular Sturm-Liouville problem on a finite interval with the eigenvalue parameter appearing linearly in the boundary conditions. It is shown that the eigenfunctions for this class of problems form a Riesz basis of the corresponding Hilbert space.

In this study, we prove that the weak eigenfunctions of the Sturm-Liouville problem with additional transmission conditions form a Riesz basis of the corresponding Hilbert space. First, the generalized solution of the Sturm-Liouville problem is defined as an element of a direct sum spaces with satisfies some integral equalities. Second, using the Riesz representation theorem these equalities are reduced to an operator-pencil equation. Finally, it is established that the eigenfunctions of the original boundary-value-transmission problem form a Riesz basis of suitable Hilbert space.

Some definitions and auxiliary results

Consider a boundary value problem for the Sturm-Liouville equations

$$-f_1''(x) + q_1(x)f_1(x) = \lambda f_1(x), \quad \text{for } x \in [0, c] \tag{2}$$

$$-f_2''(x) + q_2(x)f_2(x) = \lambda f_2(x), \quad \text{for } x \in [d, 1] \tag{3}$$

subject to boundary conditions at the end points $x = 0$ and $x = 1$, given by

$$\cos\alpha f_1(0,\lambda) + \sin\alpha \frac{\partial f_1(0,\lambda)}{\partial x} = 0, \tag{4}$$

$$f_2(1,\lambda) = 0, \tag{5}$$

and with two supplementary interface conditions at the points of interaction $x = c$ and $x = d$, given by

$$f_2(d,\lambda) - f_1(c,\lambda) = 0, \tag{6}$$

$$\frac{\partial f_2(d,\lambda)}{\partial x} - \frac{\partial f_1(c,\lambda)}{\partial x} = 0, \tag{7}$$

where $0 < c < d < 1$, the functions $q_1(x)$ and $q_2(x)$ are positive defined, measurable and Lebesgue integrable on $[0, c]$ and $[d, 1]$ respectively, λ is a complex spectral parameter, $\alpha \in [0, \pi)$.

Now, we shall define some new Hilbert spaces and give some inequalities which is needed for investigation of the considered problem (2)-(7).

Definition 1 *The direct sum of Lebesgue spaces* $\oplus L_2 := L_2[0, c] \oplus L_2[d, 1]$ *is a Hilbert space of square integrable complex valued functions on each of intervals* $[0, c]$ *and* $[d, 1]$ *with the scalar product*

$$\langle f\ ,\ g\rangle_{\oplus L_2} := \int_0^c f_1(x)\,\overline{g_1}(x)dx + \int_d^1 f_2(x)\,\overline{g_2}(x)dx \tag{8}$$

for

$$f = \begin{cases} f_1(x)\ ,\ for\ x \in [0, c] \\ f_2(x)\ ,\ for\ x \in [d, 1] \end{cases}$$

and

$$g = \begin{cases} g_1(x)\ ,\ for\ x \in [0, c] \\ g_2(x)\ ,\ for\ x \in [d, 1] \end{cases}$$

and the corresponding norm

$$\|f\|^2_{\oplus L_2} \ = \ \langle f, f\rangle_{\oplus L_2}. \tag{9}$$

Definition 2 *The direct sum of Sobolev spaces* $\oplus H^1_q := W^1_2[0,c] \ \oplus \ W^1_2[d,1]$ *is the Hilbert space consisting of all functions* $f \in L_2[0,c] \ \oplus \ L_2[d,1]$ *that have generalized derivatives* $f' \in L_2[0,c] \ \oplus \ L_2[d,1]$ *with the inner product*

$$\begin{aligned} \langle f\ ,\ g\rangle_{\oplus H^1_q} \quad := \quad & \int_0^c \Big\{ f_1'(x)\, \overline{g_1}'(x) + q_1(x) f_1(x)\, \overline{g_1}(x) \Big\} dx \\ + \quad & \int_d^1 \Big\{ f_2'(x)\, \overline{g_2}'(x) + q_2(x) f_2(x)\, \overline{g_2}(x) \Big\} dx \end{aligned} \tag{10}$$

and the corresponding norm

$$\|f\|^2_{\oplus H^1_q} \ = \ \langle f, f\rangle_{\oplus H^1_q}. \tag{11}$$

Lemma 1 *The norm (11) is equivalent to the original norm (9), i.e., there exist positive constants m and M, independent of f, such that*

$$m\, \|f\|_{\oplus L_2} < \|f\|_{\oplus H^1_q} < M\, \|f\|_{\oplus L_2}. \tag{12}$$

The following inequalities and their proofs are similar to those in Ladyzhenskaia [12].

Lemma 2 *The following inequalities are valid for any* $f \in \oplus H^1_q$.

$$|f_2(1)|^2 \le \ell\, \|f'\|^2_{\oplus L_2} + \frac{2}{\ell}\, \|f\|^2_{\oplus L_2}, \tag{13}$$

$$|f_1(0)|^2 \le \ell\, \|f'\|^2_{\oplus L_2} + \frac{2}{\ell}\, \|f\|^2_{\oplus L_2} \tag{14}$$

$$|f(d)| \quad \le \quad C_0\, \|f\|_{\oplus H^1_q} \tag{15}$$

with the constant C_0 *independent of the function* $f(x)$ *and* $d \in (0,1)$*, where* ℓ *is any positive real number is small enough.*

Main Results

The definition of a generalized solutions of the Sturm-Liouville problem (2)-(7) follows by the same procedure as in [17]. To define the generalized solution of the problem multiply equations (2)-(3) by the conjugate of an arbitrary function $\vartheta \in \oplus H^1_q$ and integrate by parts over the intervals $[0, c]$ and $[d, 1]$ and applying the boundary-transmission conditions (4)-(7), we can reduce it to the integral form

$$\langle f, \vartheta\rangle_{\oplus H^1_q} + f_2(1)\overline{\vartheta_2}(1) - \cot\alpha f_1(0)\overline{\vartheta_1}(0) = \lambda \left\{ \int_0^c f_1(x)\, \overline{\vartheta_1}(x) dx + \int_d^1 f_2(x)\, \overline{\vartheta_2}(x) dx \right\}. \tag{16}$$

Definition 3 *The element* $f(x) \in \oplus H^1_q$ *is said to be a generalized solution of the Sturm-Liouville system (2)-(7), if this element satisfy the equality* (16) *for any* $\vartheta \in \oplus H^1_q$.

For further investigation we shall introduce to the consideration the following bilinear functionals:

$$\tau_0(f,\vartheta) := f_2(1)\overline{\vartheta_2}(1) - \cot\alpha f_1(0)\overline{\vartheta_1}(0), \tag{17}$$

$$\tau_1(f,\vartheta) := \int_0^c f_1(x)\,\overline{\vartheta_1}(x)dx + \int_d^1 f_2(x)\,\overline{\vartheta_2}(x)dx. \tag{18}$$

Theorem 1 *There are linear bounded operators $L_0 : \oplus H_q^1 \to \oplus H_q^1$ and $L_1 : \oplus H_q^1 \to \oplus H_q^1$ satisfying the following representations:*

$$\tau_k(f,\vartheta) \;=\; \langle L_k f,\vartheta\rangle_{\oplus H_q^1}\ (k=0,1). \tag{19}$$

Theorem 2 *The operator $L_0 : \oplus H_q^1 \to \oplus H_q^1$ is self-adjoint and compact.*

Theorem 3 *The operator L_1 is self-adjoint, compact and positive in the Hilbert space $\oplus H_q^1$.*

Let us define the operator $R : \oplus H_q^1 \to \oplus H_q^1$ by action law

$$R(f) \;=\; f + L_0 f$$

and the operator pencil $\mathrm{T}(\lambda) : \oplus H_q^1 \to \oplus H_q^1$ by action law

$$\mathrm{T}(\lambda)f = R(f) + \lambda\, L_1.$$

Theorem 4 *The operator polynomial $\mathrm{T}(-\lambda_0)$ is positive definite for sufficiently large positive values of λ_0.*

Theorem 5 $S(\lambda) = \left(\sqrt{\mathrm{T}(-\lambda)}\,\right)^{-1} L_1 \sqrt{\mathrm{T}(-\lambda)}$ *for sufficiently large positive values of λ. We can prove that this operator is positive, self-adjoint and compact operator.*

Theorem 6 *Let $f(x)$ be any eigenfunction of the boundary-value-transmission problems (2)-(7). Then $g(x) = \mathrm{T}(\lambda_0)f$ is the eigenelement of the operator $S(\lambda_0)$.*

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