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Completeness of the Weak Eigenfunctions of one Boundary-Value-Transmission Problem

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Abstract. Recently, the basis properties and eigenfunction expansions in various function spaces of the eigenfunction of the regular boundary value problems with spectral parameter in the boundary conditions have been investigated by many mathematicans. However in different areas of applied mathematics and physics many problems arise in the form of singular boundary value problems involving transmission conditions at the interior singular points. Such problems are called boundary-value-transmission problems. For example, some boundary value problems with transmission conditions arise in heat and mass transfer problems, in vibrating string problems when the string loaded with additional point masses, in diffraction problems. It is not clear how the extend the classical methods to a problem with additional transmission conditions. Another major diffuculty lies in the completeness of the eigenfunctions, since the eigenvalues of a Sturm-Liouville problems with transmission conditions may not have an asymptotic expansion. This study devoted to the investigation of the Sturm-Liouville type boundary-value problems with supplementary transmission conditions. We introduce a new concept, so-called generalized eigenfunctions and study the completeness and basis properties of systems of generalized eigenfunctions for the problem under consideration.

Keywords : Boundary value problems, boundary conditions, weak eigenfunctions, eigenvalue, completeness.

Introduction

Several problems of physics and engineering are often stated as boundary-value problems (BVP's, for short) [5, 12, 19]. Among these BVP's, the Sturm-Liouville type problems is a typical one, since these problems plays an important role in solving various problems in mathematical physics. In the recent years, there has been growing interest in Sturm-Liouville problems with additional impulse effects (also known as transmission conditions) due to physical applications. For example, transmission problems arise in thermal conduction problems for a plate composed by materials with different characteristics piled in the thickness [19]. Recently, there have been contributions to studying the properties of Sturm-Liouville problems with transmission conditions [1, 2, 8, 13, 14, 15, 16, 17, 21, 22]

In [Walter [20], Fulton [6], Belinsky and Dauer [3, 4], Sarsenbi and Tengaeva [18], Kerimov and Poladov [10], Olğar and et. al. [16, 17]] the authors studied the basis property in various function spaces of the eigenfunction of the Sturm-Liouville problems with nonclassical boundary conditions.

The concept of generalized (weak) solutions in a Hilbert space allows the eigenvalue problem to be reduced to an operator-pencil equation (see, [11]). The fundamental work of M. V. Keldysh [9] contains adequate statements of the main problems in the spectral theory of polynomial pencils along with the first important results in this theory. Here the concepts of associated vectors, multiplicity of an eigenvalue, and multiple completeness of the eigenvectors and associated vectors were introduced. Belinskiy and Dauer [3, 4] have considered the eigenfunctions of a regular Sturm-Liouville problem on a finite interval with the eigenvalue parameter appearing linearly in the boundary conditions. It is shown that the eigenfunctions for this class of problems form a Riesz basis of the corresponding Hilbert space.

This study devoted to the investigation of the Sturm-Liouville type boundary-value problems with supplementary transmission conditions. We introduce a new concept, so-called generalized eigenfunctions and study the completeness and basis properties of systems of generalized eigenfunctions for the problem under consideration.

Namely we shall investigate the following many-interval Sturm-Liouville equation

$$-u''(x) + q(x)u(x) = \lambda u(x) \tag{1}$$

on $[-1, -\varepsilon) \cup (-\varepsilon, \varepsilon) \cup (\varepsilon, +1]$, together with boundary conditions at the end-points $x = -1$ and $x = +1$, given by

$$u(-1) = u(+1) = 0 \tag{2}$$

and with transmission conditions interior points $x = -\varepsilon$ and $x = \varepsilon$, given by

$$u(-\varepsilon^-) = u(-\varepsilon^+), \tag{3}$$

$$u'(-\varepsilon^-) - u'(-\varepsilon^+) = u(-\varepsilon), \tag{4}$$

$$u(\varepsilon^-) = u(\varepsilon^+), \tag{5}$$

$$u'(\varepsilon^-) - u'(\varepsilon^+) = u(\varepsilon) \tag{6}$$

where the function $q(x)$ is measurable, bounded and positively definite, λ is a complex spectral parameter.

Some definitions and auxiliary results

Let us list the principal function spaces appearing in the paper.

$L_2(a, b)$ is Hilbert space consisting of all Lebesgue measurable functions $u(x)$ on (a, b) for which

$$\Big(\int_a^b |u|^2 dx\Big)^{\frac{1}{2}} < \infty$$

with the scalar product given by $\langle u, v\rangle_{L_2(a,b)} := \int_a^b (u(x)\overline{v}(x)dx$.

The Sobolev space $W_2^1(a, b)$ consist of all elements $u \in L_2(a, b)$ having generalized derivative $u' \in L_2(a, b)$ with the scalar product given by

$$\langle u, v\rangle_{W_2^1(a,b)} = \int_a^b (uv + u'v')dx.$$

Now, we shall define some new Hilbert spaces and give some inequalities which is needed for investigation of the considered problem (1)-(6).

Define $\oplus L_2 := L_2(-1, -\varepsilon^-) \oplus L_2(-\varepsilon^+, \varepsilon^-) \oplus L_2(\varepsilon^+, 1)$ and $\oplus W_2^1 := W_2^1(-1, -\varepsilon^-) \oplus W_2^1(-\varepsilon^+, \varepsilon^-) \oplus W_2^1(\varepsilon^+, 1)$. The inner product in those spaces are given by

$$\langle u, v\rangle_0 = \int_{-1}^{-\varepsilon^-} u\overline{v}dx + \int_{-\varepsilon^+}^{\varepsilon^-} u\overline{v}dx + \int_{\varepsilon^+}^{1} u\overline{v}dx$$

and

$$\langle u, v\rangle_1 = \int_{-1}^{-\varepsilon^-} \Big(u\overline{v} + u_x\overline{v}_x\Big)dx + \int_{-\varepsilon^+}^{\varepsilon^-} \Big(u\overline{v} + u_x\overline{v}_x\Big)dx + \int_{\varepsilon^+}^{1} \Big(u\overline{v} + u_x\overline{v}_x\Big)dx$$

with the corresponding norms are

$$\|u\|_0 = \sqrt{\langle u, u\rangle_0}$$

and

$$\|u\|_1 = \sqrt{\langle u, u\rangle_1}$$

respectively. In the same linear space $\oplus W_2^1$ we introduce an another inner-product as

$$\langle u,v\rangle_{\oplus W^1_{2,\,q}} = \int_{-1}^{-\varepsilon}\Big(q(x)u\overline{v}+u_x\overline{v}_x\Big)dx + \int_{-\varepsilon}^{+\varepsilon}\Big(q(x)u\overline{v}+u_x\overline{v}_x\Big)dx + \int_{+\varepsilon}^{1}\Big(q(x)u\overline{v}+u_x\overline{v}_x\Big)dx.$$

Since $q(x)$ is bounded, positively defined and measurable function, there exist constant $m>0$ and $M>0$ such that

$$m\,\|u\|_1 < \|u\|_{\oplus W^1_{2,\,q}} < M\,\|u\|_1. \tag{7}$$

The following inequalities and their proofs are similar to those in Ladyzhenskaia [11].

Lemma 1 *The following inequalities are valid for any $u\in\oplus W^1_{2,\,q}$.*

$$|u(1)|^2 \le \ell\,\|u'\|^2_{\oplus L_2} + \frac{2}{\ell}\,\|u\|^2_{\oplus L_2}, \tag{8}$$

$$|u(\pm\varepsilon^{\pm})|^2 \le \ell\,\|u'\|^2_{\oplus L_2} + \frac{2}{\ell}\,\|u\|^2_{\oplus L_2}, \tag{9}$$

$$|u(-1)|^2 \le \ell\,\|u'\|^2_{\oplus L_2} + \frac{2}{\ell}\,\|u\|^2_{\oplus L_2} \tag{10}$$

$$|f(d)| \quad\le\quad C_0\,\|f\|_{\oplus H^1_q} \tag{11}$$

with the constant C_0 independent of the function $f(x)$ where ℓ is any positive real number is small enough and $d\in[-1,-\varepsilon)\cup(-\varepsilon,\varepsilon)\cup(\varepsilon,+1]$ is arbitrary number.

The definition of a generalized solutions of the Sturm-Liouville problem (1)-(5) follows by the same procedure as in [16]. Multiplying equation (1) by a conjugate to an arbitrary function $\vartheta(x)\in\oplus W_2^1$ and integrate by parts over the intervals $(-1,-\varepsilon),(-\varepsilon,+\varepsilon)$ and $(+\varepsilon,+1)$ and applying the boundary and transmission conditions (2)-(5), we can reduce it to the integral form

$$\langle u,\vartheta\rangle_{\oplus W^1_{2,\,q}} + u(-\varepsilon)\overline{\vartheta}(-\varepsilon) + u(+\varepsilon)\overline{\vartheta}(+\varepsilon) = \lambda\langle u,\vartheta\rangle_0. \tag{12}$$

Definition 1 *The element $u(x)\in\oplus W_2^1$ is said to be a generalized solution of the Sturm-Liouville system (1)-(6), if this element satisfy the equality* (12) *for any $\vartheta\in\oplus W_2^1$.*

For further investigation we shall introduce to the consideration the following bilinear functionals:

$$\kappa_0(u,\vartheta) := u(-\varepsilon)\overline{\vartheta}(-\varepsilon) + u(+\varepsilon)\overline{\vartheta}(+\varepsilon), \tag{13}$$

$$\kappa_1(u,\vartheta) := \langle u,\vartheta\rangle_0. \tag{14}$$

Theorem 1 *There are linear bounded operators $T_0:\oplus W_2^1\to\oplus W_2^1$ and $T_1:\oplus W_2^1\to\oplus W_2^1$ satisfying the following representations:*

$$\kappa_i(u,\vartheta) = \langle T_i u,\vartheta\rangle_{\oplus W_2^1}\ (i=0,1). \tag{15}$$

Lemma 2 *The operators $T_0,\ T_1:\oplus W_2^1\to\oplus W_2^1$ are self-adjoint, compact and T_1 is positive.*

The generalized eigenfunctions of the Sturm-Liouville problems (1)-(6) satisfy the operator-pencil equation in $\oplus W_2^1$,

$$\mathrm{T}(\lambda)\,\psi = 0\,, \qquad \mathrm{T}(\lambda) = \mathrm{T}_0 - \lambda\,\mathrm{T}_1, \tag{16}$$

where the operators $\mathrm{T_i}(i=0,1)$ are defined on all of $\oplus W_2^1$.

Theorem 2 *The operator polynomial* $\mathrm{T}(-\lambda_0)$ *is positive definite for sufficiently large positive values of λ_0.*

By virtue of the Theorem 2 there exists $c > 0$, such that for all real $\lambda_0 > c$ the operator polynomial $T(-\lambda_0)$ is positive definite. Moreover, we know that the operator $T(-\lambda_0)$ is also self-adjoint. Therefore there exists positive square root $\sqrt{T(-\lambda_0)}$ which is invertible. Consequently, we can introduce to consideration a new operator Z defined by

$$Z := \left(\sqrt{T(-\lambda_0)}\right)^{-1} T_1 \left(\sqrt{T(-\lambda_0)}\right)^{-1}$$

in the Hilbert space $\oplus W_2^1$.

Hence we have the following result.

Theorem 3 *The operator Z is positive, self-adjoint and compact in the Hilbert space $\oplus W_2^1$ for sufficiently large positive λ_0.*

Now, by using the well-known fact, that every bounded invertible operator transforms any orthonormal basis of a Hilbert space $\oplus W_2^1$ into Riesz basis of $\oplus W_2^1$ (see, for example, [7]), we get the following important result.

Theorem 4 *The eigenvalues of the Boundary-value-transmission problem* (1) − (6) *form a real discrete set* $\{\lambda_n\}$ *with* $\lambda_n \to +\infty$. *The system of the generalized eigenfunctions of the Boundary-value-transmission problem* (1)−(6) *forms a Riesz basis of* $\oplus W_2^1$.

Theorem 5 *The system of the generalized eigenfunctions of the boundary-value-transmission problem* (1) − (6) *forms an complete system in the Hilbert space* $\oplus L_2$.

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