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FIXED POINTS FOR WEAKLY CONTRACTIVE MAPS ON N-NORMED SPACES

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Abstract. In 1964 Gahler introduced the theory of 2-normed spaces and then 1989 n-normed spaces was first defined by Misiak. Then some researcher found new results on n-inner product spaces and n-metric spaces. In 2001 Gunawan and Mashadi studied converges and completeness in n-normed spaces and they gave a fixed point theorems for some n-Banach spaces. And in 2014 Kır and Kızıltunc defined phi-contraction map in n-Normed Spaces and proved fixed point theorems which satisfy phi-contraction. On the other hand,in 1997 Alber and Guerre-Delabriere defined weakly contractive maps and gave theorems in uniformly smooth and uniformly convex Banach spaces. Then in 2001 Rhoades extended some theorems of their work to arbitrary Banach spaces for weakly contractions. In this paper the notion of weakly contractive maps on n-Normed spaces are presented. Also some fixed point theorems for weak contractive maps are proved. Thus, we will provide a source for new studies related to weak contractions in n-normed spaces.

1. Introduction and Preliminaries

We will give some definitions and results in n-normed spaces.

Definition 1.1. *([3]) Let $n \in \mathbb{N}$ and E be a real vector space of dimension $d \geq n$. A real valued function $||., ..., .||$ on E^n satisfying the following*

$$\begin{aligned}
&n1)||x_1, ..., x_n|| = 0 \text{ if and only if } x_1, ..., x_n \text{ are linearly dependent };\\
&n2)||x_1, ..., x_n| \text{ is invariant under permutation};\\
&n3)||x_1, ..., x_{n-1}, cx_n|| = |c|||x_1, ..., x_{n-1}, x_n|| \text{ for all } c \in \mathbb{R};\\
&n4)||x_1, ..., x_{n-1}, y + z|| \leq ||x_1, ..., x_{n-1}, y|| + ||x_1, ..., x_{n-1}, z||,
\end{aligned} \tag{1}$$

is called n-norm on E and the pair $(E, ||., ..., .||)$ is called n-normed space.

Definition 1.2. *([3]) A sequence $\{x_n\}$ in a n-normed space $(E, ||., ..., .||)$ is said to be a Cauchy sequence if $\lim_{m,n\to\infty} ||x_n - x_m, x_2, ..., x_n|| = 0$ for all $x_2, ..., x_n \in E$.*

Definition 1.3. *([3]) A sequence $\{x_n\}$ in a n-normed space $(E, ||., ..., .||)$ is said to be convergent if there is a point x in E such that $\lim_{n\to\infty} ||x_n - x, x_2, ..., x_n|| = 0$ for all $x_2, ..., x_n \in E$. If $\{x_n\}$ converges to x then, we write $x_n \to x$ as $n \to \infty$.*

Definition 1.4. *([3]) A linear n-normed space is said to be complete if every Cauchy sequence is convergent to an element of E. A complete n-normed space E is called n-Banach space.*

Definition 1.5. *([4]) Let $(E, ||., ..., .||)$ be a linear n-normed space, B be a nonempty subset of E and $e \in B$ then B is said to be e-bounded if there exist some $M > 0$ such that $||e, ..., x_n|| \leq M$ for all $x_2, ..., x_n \in B$. If for all $e \in B$, B is e-bounded then B is called a bounded set.*

2. Main Results

Firstly, since each normed space produces a metric space and metric spaces are first countable topologic spaces the following definition can be given.

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Definition 2.1. *Let* $(E, ||., ..., .||)$ *be a linear* n*-normed space,* K *be a subset of* E *then the closure of* K *is* $\overline{K} = \{x \in E : \text{ there is a sequence } x_n \text{ of } K \text{ such that } x_n \to x\}$*. We say,* K *is closed if* $K = \overline{K}$*.*

Definition 2.2. *Let* $(E, ||., ..., .||)$ *be a* n*-Banach space,* K *a closed convex subset of* E*.* $T : K \to K$ *is called weakly contractive if for all* $x, y, x_2, x_3, ..., x_n \in K$*,*

$$||Tx - Ty, x_2, ..., x_n|| \leq ||x - y, x_2, ..., x_n|| - \psi(||x - y, x_2, ..., x_n||) \tag{2}$$

where $\psi : [0, \infty) \to [0, \infty)$ *is continuous and nondecreasing mapping such that* $\psi(0) = 0$*,* $\psi(t) > 0$ *for all* $t > 0$ *and* $\lim_{n\to\infty} \psi(t) = \infty$*. If* K *is bounded, then the infinity condition is not necessary.*

Lemma 2.1. *Every weakly contractive map is continuous.*

Proof. Let $(E, ||., ..., .||)$ be a n-Banach space, K a closed convex subset of E and $T : K \to K$ is weakly contractive map. Since each normed space produces a metric space and metric spaces are first countable topologic spaces, it is enough to show that T is sequentially continuous. So, assume that $\{a_n\}$ is a sequence in K and $\{a_n\} \to a \in K$ that means $||a_n - a, x_2, ..., x_n|| \to 0$ as $n \to \infty$, then

$$\begin{aligned} ||Ta_n - Ta, x_2, ..., x_n|| &\leq ||a_n - a, x_2, ..., x_n|| - \psi(||a_n - a, x_2, ..., x_n||) \\ &\leq ||a_n - a, x_2, ..., x_n|| \to 0 (n \to \infty). \end{aligned} \tag{3}$$

Hence, $\{Ta_n\} \to Ta \in K$ (since K is a closed subset of E).

□

Theorem 2.1. *Let* $(E, ||., ..., .||)$ *be a* n*-Banach space and* K *be a closed and bounded subspace of* E*. Assume that selfmap* $T : K \to K$ *is a weakly contractive mapping. Then,* T *has a unique fixed point in* K*.*

Proof. Let $b_0 \in K$ and define $b_{m+1} = Tb_m$ for all $m \in \mathbb{N}$. Since selfmap $T : K \to K$ is a weakly contractive mapping, using (2);

$$\begin{aligned} ||b_{m+1} - b_{m+2}, x_2, ..., x_n|| &= ||Tb_m - Tb_{m+1}, x_2, ..., x_n|| \\ &\leq ||b_m - b_{m+1}, x_2, ..., x_n|| - \psi(||b_m - b_{m+1}, x_2, ..., x_n||) \\ &\leq ||b_m - b_{m+1}, x_2, ..., x_n||. \end{aligned} \tag{4}$$

Therefore $\{||b_m - b_{m+1}, x_2, ..., x_n||\}$ is nonnegative nonincreasing sequence, and hence has a limit $a \geq 0$. Assume that $a > 0$. Since ψ is nondecreasing, $\psi(||b_m - b_{m+1}, x_2, ..., x_n||) \geq \psi(a) > 0$ and then from (4),

$$||b_{m+1} - b_{m+2}, x_2, ..., x_n|| \leq ||b_m - b_{m+1}, x_2, ..., x_n|| - \psi(a). \tag{5}$$

Thus, $||b_{m+N} - b_{m+N+1}, x_2, ..., x_n|| \leq ||b_m - b_{m+1}, x_2, ..., x_n|| - N\psi(a)$, this is a contradiction for N large enough. And so, $a = 0$. Now, using this fact, we shall show $\{b_m\}$ is a Cauchy sequence. Assume that $\varepsilon > 0$ is arbitrary. Then, we shall choose $N(\varepsilon)$ is such that $||b_N - b_{N+1}, x_2, ..., x_n|| \leq \min\{\frac{\varepsilon}{2}, \psi(\frac{\varepsilon}{2})\}$ for N large enough. Hence, it is enough to show that if $||x - b_N, x_2, ..., x_n|| \leq \varepsilon$ then $||Tx - b_N, x_2, ..., x_n|| \leq \varepsilon$ for this arbitrary $\varepsilon > 0$ and $N(\varepsilon)$ is such that $||b_N - b_{N+1}, x_2, ..., x_n|| \leq \min\{\frac{\varepsilon}{2}, \psi(\frac{\varepsilon}{2})\}$. Indeed,

Case 1: If $||x - b_N, x_2, ..., x_n|| \leq \frac{\varepsilon}{2}$, then

$$\begin{aligned} ||Tx - b_N, x_2, ..., x_n|| &\leq ||Tx - Tb_N, x_2, ..., x_n|| + ||Tb_N - b_N, x_2, ..., x_n|| \\ &\leq ||x - b_N, x_2, ..., x_n|| - \psi(||x - b_N, x_2, ..., x_n||) + ||b_{N+1} - b_N, x_2, ..., x_n|| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{6}$$

Case 2: If $\frac{\varepsilon}{2} < ||x-b_N, x_2, ..., x_n|| \leq \varepsilon$, then $\psi(\frac{\varepsilon}{2}) \leq \psi(||x-b_N, x_2, ..., x_n||)$ and

$$\begin{aligned} ||Tx-b_N, x_2, ..., x_n|| &\leq ||Tx-Tb_N, x_2, ..., x_n|| + ||Tb_N - b_N, x_2, ..., x_n|| \\ &\leq ||x-b_N, x_2, ..., x_n|| - \psi(||x-b_N, x_2, ..., x_n||) + ||b_{N+1} - b_N, x_2, ..., x_n|| \\ &\leq ||x-b_N, x_2, ..., x_n|| - \psi(\tfrac{\varepsilon}{2}) + \psi(\tfrac{\varepsilon}{2}) \\ &\leq \varepsilon. \end{aligned} \tag{7}$$

Thus, for all $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $||b_m - b_N, x_2, ..., x_n|| \leq \varepsilon$, for all $m > N$. Thus, $\{b_m\}$ is a Cauchy sequence in K. Since K is a closed and bounded subset of n-Banach space E, $\{b_m\}$ converges to b in K. Also the continuity of T, we obtain

$$Tb = \lim_{m\to\infty} Tb_m = \lim_{m\to\infty} b_{m+1} = b. \tag{8}$$

Therefore T has a fixed point in K. Now the uniqueness of fixed point shall be proved. Assume that $a^* \in K$ is the another fixed point of T. Using the definition of weakly contractiveness of T and the property of ψ (that is $\psi(t) > 0$ for $t > 0$),

$$|a-a^*, x_2, ..., x_n|| = |Ta-Ta^*, x_2, ..., x_n|| \leq |a-a^*, x_2, ..., x_n|| - \psi(|a-a^*, x_2, ..., x_n||) < |a-a^*, x_2, ..., x_n|| \tag{9}$$

This implies $|a-a^*, x_2, ..., x_n|| = 0$ and thus we get $a = a^*$ in K. So, the fixed point is unique. □

Corollary 2.1. *Let $(E, ||., ..., .||)$ be a n-Banach space and K be a closed and bounded subspace of E. Assume that there exists $k \in [0, 1)$ to provide the selfmap $T : K \to K$ is satisfying the following inequality,*

$$||Tx - Ty, x_2, ..., x_n|| \leq k||x-y, x_2, ..., x_n|| \tag{10}$$

for all $x, y, x_2, x_3, ..., x_n \in K$,. Then, T has a unique fixed point in K.

Proof. If we choose $\psi : [0, \infty) \to [0, \infty), \psi(x) = (1-k)x$ for using $k \in [0, 1)$, then ψ is continuous and nondecreasing mapping such that $\psi(0) = 0$, $\psi(t) > 0$ for all $t > 0$. Thus, the all hypothesis of Theorem 2.1 are satisfied and so T has a unique fixed point in K. □

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