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SOME COUPLED FIXED POINT THEOREMS FOR MAPPINGS IN PARTIALLY ORDERED G-METRIC SPACES BY USING A RATIONAL TYPE CONTRACTIVE CONDITION

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Abstract. Coupled fixed point results for nonlinear contraction mappings having a mixed monotone property in a partially ordered G-metric space due to Choudhury and Maity are extended and unified. In this paper, some corresponding coupled fixed point theorems are obtained in partially ordered G-metric spaces by employing a rational type contractive condition. These results generalize and extend some recently announced results in the literature.

1. Introduction and Preliminaries

The notion of coupled fixed points was introduced by Chang and Ma [6]. Since then, the concept has been of interest to many researchers in metrical fixed point theory.

Bhaskar and Lakshmikantham [3] introduced the concept of a coupled fixed point and the mixed monotone property. And they established coupled fixed point theorems in a metric space endowed with partial order by employing the following contractivity condition :

For a mapping $T : X \times X \to X$, there exists $k \in (0, 1)$ such that $d(T(x, y), T(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)]$, for all $x, y, u, v \in X, x \geq u, y \leq v$.

Thus, they proved some coupled fixed point theorems for mappings which satisfy the mixed monotone property and gave some applications in the existence and uniqueness of a solution for a periodic boundary value problem.

Harjani et al [11] established some fixed point theorems in partially ordered metric space setting by using a contractive condition of rational type. That is, for a mapping $T : X \to X$, there exists some $\alpha, \beta \in [0, 1]$, with $\alpha + \beta < 1$, such that $d(Tx, Ty) \leq \alpha \frac{d(x,Tx).d(y,Ty)}{d(x,y)} + \beta d(x, y)$, for all $x, y \in X, x \neq y$. Then, motivated by the work of Harjani et al [11] ,Ciric et al [10] obtained some corresponding coupled fixed point theorems in partially ordered metric spaces by employing a rational type contractive condition.

Mustafa and Sims [13, 14] introduced a new concept of generalized metric spaces, called G-metric spaces. In such spaces every triplet of elements is assigned to a non-negative real number. Based on the notion of G-metric spaces, Mustafa et al. [15] established fixed point theorems in G-metric spaces. Afterward, many fixed point results were proved in this space [1, 2, 9, 16, 17, 18, 19].

Recently, Choudhury and Maity [8] studied necessary conditions for existence of coupled fixed point in partially ordered G-metric spaces.

Then, Chakrabarti [5] developed a coupled fixed point theorem using a rational type, nonlinear contractive condition in a partially ordered complete G-metric space. The condition is similar to the rational type contractive condition of Ciric et al [10] and may be considered as a generalization of the condition given in [10].

In this paper we obtain some corresponding coupled fixed point theorems in partially ordered complete G-metric spaces by employing a rational type contractive condition as different from the one in [5]. Indeed, the restriction on α and β is that $8\alpha + \beta < 1$ in [5]. However, we choose $\alpha + \beta < 1$ and use the fact that $G(x_{n-1}, x_n, x_n) = G(x_n, x_n, x_{n-1})$ in our paper. Thus, one can see that it is possible to relax the condition on α and β. As a result, our results generalize and extend some recently announced results in the literature.

Throughout this article, $(X, \preceq)$ denotes a partially ordered set with partial order $\preceq$.

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If $(X, \preceq)$ is a partially ordered set, a mapping $f : X \to X$ is said to be non-decreasing (non-increasing) if for all $x, y \in X, x \preceq y$ implies $f(x) \preceq f(y)$ ($f(y) \preceq f(x)$, respectively).

Definition 1.1. *(See [3]) Let $(X, \preceq)$ be a partial ordered set. A mapping $T : X \times X \to X$ is said to has the a mixed monotone property if T is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for any $x, y \in X$.*

$$x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow T(x_1, y) \preceq T(x_2, y) \tag{1}$$

and

$$y_1, y_2 \in X, y_1 \preceq y_2 \Longrightarrow T(x, y_1) \succeq T(x, y_2). \tag{2}$$

Definition 1.2. *(See [3]) An element $(x, y) \in X \times X$ is called a coupled fixed point of mapping $T : X \times X \to X$ if*

$$x = T(x, y) \text{ and } y = T(y, x). \tag{3}$$

Definition 1.3. *(See [12]) Let $(X, \preceq)$ be a partial ordered set and $T : X \times X \to X$ and $g : X \to X$. We say T has the mixed g-monotone property if T is monotone g-nondecreasing in its first argument and is monotone g-nonincreasing in its second argument. That is, for all $x, y \in X$,*

$$x_1, x_2 \in X, gx_1 \preceq gx_2 \Rightarrow T(x_1, y) \preceq T(x_2, y) \tag{4}$$

and

$$y_1, y_2 \in X, gy_1 \preceq gy_2 \Longrightarrow T(x, y_1) \succeq T(x, y_2). \tag{5}$$

Definition 1.4. *(See [12]) An element $(x, y) \in X \times X$ is called a coupled coincidence point of the maps $T : X \times X \to X$ and $g : X \to X$ if $T(x, y) = gx$ and $T(y, x) = gy$.*

Definition 1.5. *(See [12]) The maps $T : X \times X \to X$ and $g : X \to X$ are said to be commutative if $g(T(x, y)) = T(gx, gy)$.*

Consistent with Mustafa and Sims [13, 14], the following definitions and results will be needed in the sequel.

Definition 1.6. *(See [14]) Let X be a non-empty set, $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties :*
(G1) $G(x, y, z) = 0$ *if $x = y = z$,*
(G2) $0 < G(x, x, y)$ *for all $x, y \in X$ with $x \neq y$,*
(G3) $G(x, x, y) \leq G(x, y, z)$ *for all $x, y, z \in X$ with $y \neq z$,*
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ *(symmetry in all three variables),*
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *(rectangle inequality) for all $x, y, z, a \in X$.*
Then the function G is called a generalized metric, or, more specifically, a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 1.7. *(See [14]) Let X be a G-metric space and let $\{x_n\}$ be a sequence of points of X, a point $x \in X$ is said to be the limit of a sequence $\{x_n\}$ if $G(x, x_n, x_m) \to 0$ as $n, m \to \infty$ and sequence $\{x_n\}$ is said to be G-convergent to x.*

From this definition, we obtain that if $x_n \to x$ in a G-metric space X, then for any $\varepsilon > 0$ there exists a positive integer N such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$.

It has been shown in [14] that the G-metric induces a Hausdorff topology and the convergence described in the above definition is relative to this topology. So, a sequence can converge at the most to one point.

Definition 1.8. *(See [14]) Let X be a G-metric space, a sequence $\{x_n\}$ is called G-Cauchy if for every $\varepsilon > 0$ there is a positive integer N such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq N$, that is, if $G(x_n, x_m, x_l) \to 0$, as $n, m, l \to \infty$.*

We next state the following lemmas.

Lemma 1.1. *(See [14]) Let (X, G) be a G-metric space. The following are equivalent:*
(1) $\{x_n\}$ *is G-convergent to x,*
(2) $G(x_n, x_n, x) \to 0$ *as* $n \to +\infty$,
(3) $G(x_n, x, x) \to 0$ *as* $n \to +\infty$,
(4) $G(x_n, x_m, x) \to 0$ *as* $n, m \to +\infty$.

Lemma 1.2. *(See [14]) If X is a G-metric space, then the following are equivalent:*

(a) The sequence $\{x_n\}$ is G-Cauchy.
(b) For every $\varepsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.

Lemma 1.3. *(See [14]) If X is a G-metric space then $G(x, y, y) \leq 2G(y, x, x)$ for all $x, y \in X$.*

Definition 1.9. *(See [14]) Let $(X, G), (X^1, G^1)$ be two generalized metric spaces. A mapping $f : X \to X^1$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G^1- convergent to $f(x)$.*

Definition 1.10. *(See [14]) A G-metric space X is called a symmetric G-metric space if*

$$G(x, y, y) = G(y, x, x) \text{ for all } x, y \in X. \tag{6}$$

Definition 1.11. *(See [14]) A G-metric space X is said to be G-complete metric space if every G-Cauchy sequence in X is convergent in X.*

Definition 1.12. *(See [14]) Let X be a G-metric space. A mapping $F : X \times X \to X$ is said to be continuous if for any two G-convergent sequences $\{x_n\}$ and $\{y_n\}$ converging to x and y, respectively, $\{F(x_n, y_n)\}$ is G-convergent to $F(x, y)$.*

2. Main Results

Now, we examine the appropriate conditions for two maps $T : X \times X \to X$ and $g : X \to X$ have a coupled coincidence point.

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete $G-$metric on X, $T : X \times X \to X$ and $g : X \to X$ are continuous mappings on X such that T has the mixed $g-$monotone property. Suppose that $T(X \times X) \subseteq g(X)$, g commutes with T, and, for some $\alpha, \beta \in [0, 1)$, for all $x, y, z, u, v, w \in X$ for which $w \preceq u \preceq x$ and $y \preceq v \preceq z$, where either $x \neq w$ or $y \neq z$, and, for $gw \preceq gu \preceq gx$ and $gy \preceq gv \preceq gz$, where either $gx \neq gw$ or $gy \neq gz$, we have*

$$G(T(x, y), T(u, v), T(w, z)) \preceq \alpha \frac{G(gx, T(x, y), T(x, y)).G(gw, T(w, z), T(w, z))}{G(gx, gu, gw)} + \beta G(gx, gu, gw), \alpha + \beta < 1. \tag{7}$$

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq T(x_0, y_0)$ and $gy_0 \succeq T(y_0, x_0)$ then T and g have a coupled coincidence point $(x^*, y^*) \in X \times X$; that is (x^*, y^*) satisfies $gx^* = T(x^*, y^*)$, $gy^* = T(y^*, x^*)$.

Proof. Since $T(X \times X) \subseteq g(X)$ we can choose $x_1, y_1 \in X$ such that $gx_1 = T(x_0, y_0), gy_1 = T(y_0, x_0)$. Similarly, points $x_2, y_2 \in X$ can be found such that $gx_2 = T(x_1, y_2), gy_2 = T(y_1, x_1)$. Due to the mixed $g-$ monotone property of T, we have $gx_0 \preceq gx_1 \preceq gx_2$ and $gy_2 \preceq gy_1 \preceq gy_0$. In general, it can be shown that, [12] for $n \geq 0$,

$$\begin{aligned} gx_n = T(x_{n-1}, y_{n-1}) \preceq gx_{n+1} = T(x_n, y_n), \\ gy_{n+1} = T(y_n, x_n) \preceq gy_n = T(x_{n-1}, y_{n-1}). \end{aligned} \tag{8}$$

If, for any $n \in \mathbb{N}$, $gx_n = gx_{n+1}$ and $gy_n = gy_{n+1}$, then (x_n, y_n) is a coupled coincidence point of T and g.
So, we may assume that, for all $n \in \mathbb{N}$, $gx_n \neq gx_{n+1}$ and $gy_n \neq gy_{n+1}$.
By condition (7),

$$\begin{aligned} G(gx_{n+1}, gx_{n+1}, gx_n) &= G(T(x_n, y_n), T(x_n, y_n), T(x_{n-1}, y_{n-1})) \\ &\leq \alpha \tfrac{G(gx_n, T(x_n, y_n), T(x_n, y_n)).G(gx_{n-1}, T(x_{n-1}, y_{n-1}), T(x_{n-1}, y_{n-1}))}{G(gx_n, gx_n, gx_{n-1})} \\ &\quad + \beta G(gx_n, gx_n, gx_{n-1}) \\ &= \alpha \tfrac{G(gx_n, gx_{n+1}, gx_{n+1}).G(gx_{n-1}, gx_n, gx_n)}{G(gx_n, gx_n, gx_{n-1})} + \beta G(gx_n, gx_n, gx_{n-1}) \\ &= \alpha G(gx_n, gx_{n+1}, gx_{n+1}) + \beta G(gx_n, gx_n, gx_{n-1}), \end{aligned} \tag{9}$$

from it follows that

$$G(gx_{n+1},gx_{n+1},gx_n) \le (\frac{\beta}{1-\alpha})G(gx_n,gx_n,gx_{n-1}). \tag{10}$$

Similarly, we have by (7) again that

$$\begin{aligned} G(gy_n,gy_{n+1},gy_{n+1}) &= G(T(y_{n-1},x_{n-1}),T(y_n,x_n),T(y_n,x_n)) \\ &\le \alpha\frac{G(gy_{n-1},T(y_{n-1},x_{n-1}),T(y_{n-1},x_{n-1})).G(gy_n,T(y_n,x_n),T(y_n,x_n))}{G(gy_{n-1},gy_n,gy_n)} + \beta G(gy_{n-1},gy_n,gy_n) \\ &= \alpha\frac{G(gy_{n-1},gy_n,gy_n).G(gy_n,gy_{n+1},gy_{n+1})}{G(gy_{n-1},gy_n,gy_n)} + \beta G(gy_{n-1},gy_n,gy_n) \\ &= \alpha G(gy_n,gy_{n+1},gy_{n+1}) + \beta G(gy_{n-1},gy_n,gy_n), \end{aligned} \tag{11}$$

from which it follows that

$$G(gy_n,gy_{n+1},gy_{n+1}) \le (\frac{\beta}{1-\alpha})G(gy_{n-1},gy_n,gy_n). \tag{12}$$

We obtain from the sum of (10) and (12) that

$$G(gx_n,gx_{n+1},gx_{n+1}) + G(gy_n,gy_{n+1},gy_{n+1}) \le (\frac{\beta}{1-\alpha})[G(gx_n,gx_n,gx_{n-1}) + G(gy_n,gy_n,gy_{n-1})]. \tag{13}$$

Let $\delta_n = G(gx_n,gx_{n+1},gx_{n+1}) + G(gy_n,gy_{n+1},gy_{n+1})$ and $\lambda = \frac{\beta}{1-\alpha}$. Then we have from (13) that

$$\delta_n \le \lambda\delta_{n-1} \le \lambda^2\delta_{n-2} \le ... \le \lambda^n\delta_0. \tag{14}$$

From our assumption that $\delta_0 > 0$, for each $r \in \mathbb{N}$, we obtain by (14) and the repeated application of the rectangle inequality, that

$$\begin{aligned} G(gx_n,gx_{n+r},gx_{n+r}) + G(gy_n,gy_{n+r},gy_{n+r}) &\le [G(gx_n,gx_{n+1},gx_{n+1}) + G(gx_{n+1},gx_{n+2},gx_{n+2}) + ... \\ &\quad +G(gx_{n+r-1},gx_{n+r},gx_{n+r})] + [G(gy_n,gy_{n+1},gy_{n+1}) \\ &\quad +G(gy_{n+1},gy_{n+2},gy_{n+2}) + ... + G(gy_{n+r-1},gy_{n+r},gy_{n+r})] \\ &= [G(gx_n,gx_{n+1},gx_{n+1}) + G(gy_n,gy_{n+1},gy_{n+1})] \\ &\quad +[G(gx_{n+1},gx_{n+2},gx_{n+2}) + G(gy_{n+1},gy_{n+2},gy_{n+2})] \\ &\quad +... + [G(gx_{n+r-1},gx_{n+r},gx_{n+r}) + G(gy_{n+r-1},gy_{n+r},gy_{n+r})] \\ &\le \delta_n + \delta_{n+1} + \delta_{n+2} + ... + \delta_{n+r-1} \\ &\le \lambda^n\delta_0 + \lambda^{n+1}\delta_0 + ... + \lambda^{n+r-1}\delta_0 \\ &= \frac{\lambda^n(1-\lambda^r)}{1-\lambda}\delta_0 \longrightarrow 0 \text{ as } n,r \longrightarrow \infty. \end{aligned} \tag{15}$$

Therefore $(gx_n),(gy_n)$ are Cauchy sequences in (X,G). Since (X,G) is complete metric space, there exist $x^*,y^* \in X$ such that $\lim gx_n = x^*$ and $\lim gy_n = y^*$. We now show that (x^*,y^*) is a coupled coincidence point of T and g. Since, T and g commute, we have

$$\begin{aligned} g(gx_{n+1}) &= g(T(x_n,y_n)) = T(gx_n,gy_n), \\ g(gy_{n+1}) &= g(T(y_n,x_n)) = T(gy_n,gx_n). \end{aligned} \tag{16}$$

Taking limit as $n \to \infty$ in (16) and noting that T and g are, respectively, continuous on $X \times X$ and X, we get

$$\begin{aligned} gx^* &= \lim g(gx_{n+1}) = \lim g(T(x_n,y_n)) = \lim T(gx_n,gy_n), \\ gy^* &= \lim g(gy_{n+1}) = \lim g(T(y_n,x_n)) = \lim T(gy_n,gx_n). \end{aligned} \tag{17}$$

By using G as metric is continuous in all its variables, we obtain

$$\begin{aligned} G(T(x^*,y^*),gx^*,gx^*) &= G(\lim T(gx_n,gy_n),gx^*,gx^*) \\ &= G(gx^*,gx^*,gx^*) \\ &= 0. \end{aligned} \tag{18}$$

So $gx^* = T(x^*,y^*)$. Similarly, we show that $gy^* = T(y^*,x^*)$. Thus, (x^*,y^*) is a coupled coincidence point of T and g. □

Theorem 2.2. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete $G-$metric on X and $T : X \times X \to X$ be a continuous mapping which has the mixed monotone property such that, for some $\alpha, \beta \in [0, 1)$, for all $x, y, z, u, v, w \in X$ for which $w \preceq u \preceq x$ and $y \preceq v \preceq z$ where either $x \neq w$ or $y \neq z$, we have*
(19)
$$G(T(x,y),T(u,v),T(w,z)) \preceq \alpha \frac{G(x,T(x,y),T(x,y)).G(w,T(w,z),T(w,z))}{G(x,u,w)} + \beta G(x,u,w), \alpha+\beta<1.$$
If there exist $x_0, y_0 \in X$ such that $x_0 \preceq T(x_0, y_0)$ and $y_0 \succeq T(y_0, x_0)$ then T has a coupled fixed point.

Proof. If, in Theorem 2.1 one sets $g = I$, the identity map, then one obtains Theorem 2.2. □

The W map was introduced by Chen [7] and a subclass of W functions was defined by Chakrabarti in [4].

Definition 2.1. *(See [4]) We call $\varphi : R^+ \to R^+$ a function of class W_β if there is a β such that $0 < \beta < 1$, and the following conditions are satisfied:*
(1) $\varphi(t) \leq \beta t$ for all $t > 0$ and $\varphi(0) = 0$,
(2) $\lim_{t_n \to t} inf \varphi(t_n) \leq \beta t$ for all $t > 0$.

The following theorems which are generalizations of respectively Theorem 2.1 and Theorem 2.2 are given by using Definition 2.1.

Corollary 2.1. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete $G-$metric on X, $T : X \times X \to X$ and $g : X \to X$ are continuous mappings on X such that T has the mixed $g-$ monotone property. Suppose that $T(X \times X) \subseteq g(X)$ and g commutes with T. For some given $\alpha, \beta \in [0, 1)$, let $\phi \in W_\alpha$ and $\psi \in W_\beta$ where $\alpha + \beta < 1$. For all $x, y, z, u, v, w \in X$ for which $w \preceq u \preceq x$ and $y \preceq v \preceq z$ where either $x \neq w$ or $y \neq z$, and, for $gw \preceq gu \preceq gx$ and $gy \preceq gv \preceq gz$ where either $gx \neq gw$ or $gy \neq gz$, we have*

(20)
$$G(T(x,y),T(u,v),T(w,z)) \preceq \phi(\frac{G(gx,T(x,y),T(x,y)).G(gw,T(w,z),T(w,z))}{G(gx,gu,gw)})+\psi(G(gx,gu,gw)), \alpha+\beta<1.$$
If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq T(x_0, y_0)$ and $gy_0 \succeq T(y_0, x_0)$ then T and g have a coupled coincidence point $(x^*, y^*) \in X \times X$; that is (x^*, y^*) satisfies $gx^* = T(x^*, y^*), gy^* = T(y^*, x^*)$.

Proof. Since $\phi \in W_\alpha$ and $\psi \in W_\beta$, using Definition 2.1 we obtain that $\phi(t) < \alpha t$ and $\psi(t) < \beta t$ for all $t > 0$. Then inequality (20) becomes equivalent to inequality (7) of Theorem 2.1. Thus the proof is similar to the proof of Theorem 2.1. □

Corollary 2.2. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete $G-$metric on X and $T : X \times X \to X$ be a continuous mapping which has the mixed monotone property. For some given $\alpha, \beta \in [0, 1)$, let $\phi \in W_\alpha$ and $\psi \in W_\beta$ where $\alpha + \beta < 1$. Suppose also that for all $x, y, z, u, v, w \in X$ for which $w \preceq u \preceq x$ and $y \preceq v \preceq z$ where either $x \neq w$ or $y \neq z$, we have*

(21)
$$G(T(x,y),T(u,v),T(w,z)) \preceq \phi(\frac{G(x,T(x,y),T(x,y)).G(w,T(w,z),T(w,z))}{G(x,u,w)}) + \psi(G(x,u,w)).$$

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq T(x_0, y_0)$ and $y_0 \succeq T(y_0, x_0)$ then T has a coupled fixed point.

Proof. Since $\phi \in W_\alpha$ and $\psi \in W_\beta$, using Definition 2.1 we obtain that $\phi(t) < \alpha t$ and $\psi(t) < \beta t$ for all $t > 0$. Then inequality (21) becomes equivalent to inequality (19) of Theorem 2.2. Thus the proof is similar to the proof of Theorem 2.2. □

3. Consequences

In this section we give an improved version of the main theorem in [5]. Indeed, we choose $\alpha + \beta < 1$ and use the fact that $G(x_{n-1}, x_n, x_n) = G(x_n, x_n, x_{n-1})$. Thus, one can see that it is possible to relax the condition on α and β. As a result, our results generalize and extend some recently announced results in the literature.

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete $G-$metric on X and $T : X \times X \to X$ be a continuous mapping which has the mixed monotone property such that, for some $\alpha, \beta \in [0, 1)$, for all $x, y, z, u, v, w \in X$ for which $w \preceq u \preceq x$ and $y \preceq v \preceq z$ where either $x \neq w$ or $y \neq z$, we have*

(22)
$$G(T(x,y),T(u,v),T(w,z)) \preceq \alpha\frac{G(x,x,T(x,y)).G(u,u,T(u,v))G(w,w,T(w,z))}{[G(x,u,w)]^2}+\beta G(x,u,w), \alpha+\beta<1.$$

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq T(x_0, y_0)$ and $y_0 \succeq T(y_0, x_0)$ then T has a coupled fixed point.

Proof. Since there exist $x_0, y_0 \in X$ such that $x_0 \preceq T(x_0, y_0)$ and $y_0 \succeq T(y_0, x_0)$, we can choose $x_1 = T(x_0, y_0) \succeq x_0$ and $y_1 = T(y_0, x_0) \preceq y_0$.
If we define $x_2 = T(x_1, y_1)$, $y_2 = T(y_1, x_1)$ and

$$\begin{aligned} T^2(x_0,y_0) = T(T(x_0,y_0),T(y_0,x_0)) = T(x_1,y_1) = x_2, \\ T^2(y_0,x_0) = T(T(y_0,x_0),T(x_0,y_0)) = T(y_1,x_1) = y_2. \end{aligned} \tag{23}$$

From the mixed monotone property of T, we get

$$\begin{aligned} x_2 = T^2(x_0,y_0) = T(x_1,y_1) \succeq T(x_0,y_1) \succeq T(x_0,y_0) = x_1 \succeq x_0, \\ y_2 = T^2(y_0,x_0) = T(y_1,x_1) \preceq T(y_1,x_0) \preceq T(y_0,x_0) = y_1 \preceq y_0. \end{aligned} \tag{24}$$

Iteratively, for all $n \in \mathbb{N}$, if we define

$$\begin{aligned} x_{n+1} = T^{n+1}(x_0,y_0) = T(T^n(x_0,y_0),T^n(y_0,x_0)), \\ y_{n+1} = T^{n+1}(y_0,x_0) = T(T^n(y_0,x_0),T^n(x_0,y_0)), \end{aligned} \tag{25}$$

then $(x_n), (y_n)$ are sequences in X such that

$$x_0 \preceq x_1 \preceq x_2 \preceq \ldots \preceq x_n \preceq x_{n+1} \preceq \ldots \tag{26}$$

and

$$y_0 \succeq y_1 \succeq y_2 \succeq \ldots \succeq y_n \succeq y_{n+1} \succeq \ldots \tag{27}$$

If for any $n \in \mathbb{N}$, $x_n = x_{n+1}$ and $y_n = y_{n+1}$ then (x_n, y_n) is a coupled fixed point of T.
So, we may assume that for all $n \in \mathbb{N}$, $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1}$.
By condition (22),

$$\begin{aligned} G(x_{n+1},x_n,x_n) &= G(T(x_n,y_n),T(x_{n-1},y_{n-1}),T(x_{n-1},y_{n-1})) \\ &\leq \alpha\tfrac{G(x_n,x_n,T(x_n,y_n)).G(x_{n-1},x_{n-1},T(x_{n-1},y_{n-1})).G(x_{n-1},x_{n-1},T(x_{n-1},y_{n-1}))}{[G(x_n,x_{n-1},x_{n-1})]^2} \\ &\quad +\beta G(x_n,x_{n-1},x_{n-1}) \\ &= \alpha\tfrac{G(x_n,x_n,x_{n+1}).G(x_{n-1},x_{n-1},x_n).G(x_{n-1},x_{n-1},x_n)}{[G(x_n,x_{n-1},x_{n-1})]^2} + \beta G(x_n,x_{n-1},x_{n-1}) \\ &= \alpha G(x_n,x_n,x_{n+1}) + \beta G(x_n,x_{n-1},x_{n-1}), \end{aligned} \tag{28}$$

and so

$$G(x_{n+1},x_n,x_n) \leq (\frac{\beta}{1-\alpha})G(x_n,x_{n-1},x_{n-1}). \tag{29}$$

Similarly, we have by (22) again that

(30)
$$\begin{aligned} G(y_n,y_n,y_{n+1}) &= G(T(y_{n-1},x_{n-1}),T(y_{n-1},x_{n-1}),T(y_n,x_n)) \\ &\leq \alpha\tfrac{G(y_{n-1},y_{n-1},T(y_{n-1},x_{n-1})).G(y_{n-1},y_{n-1},T(y_{n-1},x_{n-1})).G(y_n,y_n,T(y_n,x_n))}{[G(y_{n-1},y_{n-1},y_n)]^2} + \beta G(y_{n-1},y_{n-1},y_n) \\ &= \alpha\tfrac{G(y_{n-1},y_{n-1},y_n).G(y_{n-1},y_{n-1},y_n).G(y_n,y_n,y_{n+1})}{[G(y_{n-1},y_{n-1},y_n)]^2} + \beta G(y_{n-1},y_{n-1},y_n) \\ &= \alpha G(y_n,y_n,y_{n+1}) + \beta G(y_{n-1},y_{n-1},y_n), \end{aligned}$$

from which it follows that

$$G(y_{n+1},y_n,y_n) \leq (\frac{\beta}{1-\alpha})G(y_n,y_{n-1},y_{n-1}). \tag{31}$$

We obtain from the sum of (29) and (31) that

$$G(x_n,x_n,x_{n+1}) + G(y_n,y_n,y_{n+1}) \leq (\frac{\beta}{1-\alpha})[G(x_n,x_{n-1},x_{n-1}) + G(y_n,y_{n-1},y_{n-1})]. \tag{32}$$

Let $\delta_n = G(x_{n+1}, x_n, x_n) + G(y_{n+1}, y_n, y_n)$ and $\lambda = \frac{\beta}{1-\alpha}$. Then we have from (32) that

$$\delta_n \leq \lambda\delta_{n-1} \leq \lambda^2\delta_{n-2} \leq ... \leq \lambda^n\delta_0. \tag{33}$$

From our assumption that $\delta_0 > 0$, for each $r \in \mathbb{N}$, we obtain by (33) and the repeated application of the rectangle inequality, that

(34)

$$\begin{aligned} G(x_{n+r}, x_n, x_n) + G(y_{n+r}, y_n, y_n) &\leq [G(x_{n+r}, x_{n+r-1}, x_{n+r-1}) + G(x_{n+r-1}, x_{n+r-2}, x_{n+r-2}) + ... \\ &\quad + G(x_{n+1}, x_n, x_n)] + [G(y_{n+r}, y_{n+r-1}, y_{n+r-1}) \\ &\quad + G(y_{n+r-1}, y_{n+r-2}, y_{n+r-2}) + ... + G(y_{n+1}, y_n, y_n)] \\ &= [G(x_{n+1}, x_n, x_n) + G(y_{n+1}, y_n, y_n)] \\ &\quad + [G(x_{n+2}, x_{n+1}, x_{n+1}) + G(y_{n+2}, y_{n+1}, y_{n+1})] \\ &\quad + ... + [G(x_{n+r}, x_{n+r-1}, x_{n+r-1}) + G(y_{n+r}, y_{n+r-1}, y_{n+r-1})] \\ &\leq \delta_n + \delta_{n+1} + \delta_{n+2} + ... + \delta_{n+r-1} \\ &\leq \lambda^n\delta_0 + \lambda^{n+1}\delta_0 + ... + \lambda^{n+r-1}\delta_0 \\ &= \frac{\lambda^n(1-\lambda^r)}{1-\lambda}\delta_0 \longrightarrow 0 \text{ as } n, r \longrightarrow \infty. \end{aligned}$$

Therefore $(x_n), (y_n)$ are Cauchy sequences in (X, G). Since (X, G) is complete metric space, there exist $x^*, y^* \in X$ such that $\lim x_n = x^*$ and $\lim y_n = y^*$. We now show that (x^*, y^*) is a coupled fixed point of T. Let $\varepsilon > 0$. The continuity of T at (x^*, y^*) implies that, for a given $\frac{\varepsilon}{3} > 0$, there exists a $\delta > 0$, such that $G(x^*, u, u) + G(y^*, v, v) < \delta$ implies that $G(T(x^*, y^*), T(u, v), T(u, v)) < \frac{\varepsilon}{3}$. Since $(x_n) \to x^*$ and $(y_n) \to y^*$, for $\xi = \min(\frac{\varepsilon}{3}, \frac{\delta}{2}) > 0$, there exist n_0, m_0, such that for $n \geq n_0$, $m \geq m_0$, we have $G(x_n, x_n, x^*) < \xi$ and $G(y_m, y_m, y^*) < \xi$. Therefore, for $n \in \mathbb{N}$, $n > \max\{n_0, m_0\}$.

(35)

$$\begin{aligned} G(T(x^*, y^*), x^*, x^*) &\leq G(T(x^*, y^*), x_{n+1}, x_{n+1}) + G(x_{n+1}, x^*, x^*) \\ &\leq G(T(x^*, y^*), T(x_n, y_n), T(x_n, y_n)) + G(x^*, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+1}, x^*) \\ &< \frac{\varepsilon}{3} + \xi + \xi \\ &\leq \varepsilon \end{aligned}$$

from which it follows that $T(x^*, y^*) = x^*$. In a similar manner, we can show that $T(y^*, x^*) = y^*$. Hence (x^*, y^*) is a coupled fixed point of T. This complete the proof. □

Example 3.1. *Let $X = [0, \infty)$ and consider the function $G : X \times X \times X \to R^+$ defined by*

$$G(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{if otherwise.} \end{cases} \tag{36}$$

It is obvious that (X, G) is G-metric space. We define a partial order $\preceq$ on X by the following; for any $x, y \in X$, $x \preceq y$ if $x \leq y$. And let $T : X \times X \to X$ be defined by

$$T(x, y) = \begin{cases} 0 & \text{if } x \preceq y, \\ 2 & \text{if otherwise.} \end{cases} \tag{37}$$

Suppose $x, y, u, v, w, z \in X$ satisfy $w \preceq u \preceq x \preceq y \preceq v \preceq z$ with nonzero w and x, u, w are different from eachother. So the left side of (22) is $G(0, 0, 0) = 0$. The right side of (22)

$$\begin{aligned} &= \alpha\frac{G(x,x,0).G(u,u,0).G(w,w,0)}{[G(x,u,w)]^2} + \beta G(x, u, w) \\ &= \alpha\frac{xuw}{x^2} + \beta x \\ &= \frac{\alpha uw + \beta x^2}{x} > 0 \text{ with } \alpha = \tfrac{1}{3}, \beta = \tfrac{1}{2}. \end{aligned} \tag{38}$$

If $x_0 = 0$ and $y_0 = 2$ then $x_0 \preceq T(x_0, y_0)$ and $y_0 \succeq T(y_0, x_0)$. So all conditions of Theorem 3.1 are satisfied. Easily we can see that $(0, 2)$ and $(2, 0)$ are coupled fixed points of T.

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