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Decomposition of a Fourth-Order Linear Time-Varying System

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Abstract – In this presentation, conditions and explicit formulas for the realization of a relaxed fourth-order linear time-varying system as a cascade connection of two commutative first and third-order systems are given. The results are supported by an example.

Keywords *– Differential equation, cascade-connected system, equivalent circuit, decomposition, fourth-order system*

I. INTRODUCTION

It is a common method to design systems in the form of series (or cascade) connection of subsystems. In many cases, the order of connection of subsystems may lead positive results when the system properties such as sensitivity, stability, robustness are considered. In the cases where the change of the order of subsystems does not affect input-output relational function of the system (commutativity), the optimal order should be used in regard to improve the mentioned properties.

The first publication about the commutativity was studied in 1977 by Marshall for the first-order continuous time-varying linear systems [1]. Then, commutativity conditions of second-order [2] and third & fourth-order [3] continuous time-varying linear systems were obtained in 1982 and 1985 respectively by Koksal. After a long time, commutativity conditions of fifth-order continuous time-varying linear systems [4] were studied in 2011.

Decomposition of time-varying linear systems plays an important role toward modeling, analysing, solving real engineering problems and improving the stability of systems. In 2016, M. E. Koksal proved the necessary and sufficient conditions for the decomposition of a second-order linear time-varying system into its two cascade connected first-order commutative pairs in [5]. Then, in [6] he also studied the decomposition of a third-order linear time-varying system into its cascade connected second and first-order commutative pairs, explicit results of the subsystem, application alongside with simulation are presented. Finally, conditions end explicit formulas for the realization of a relaxed fourth-order linear time-varying system as a cascade connection of two commutative second-order systems were presented in [7].

In this presentation, conditions and explicit formulas for the realization of a relaxed fourth-order linear time-varying system as a cascade connection of commutative third and first-order systems are given. The results are illustrated by an example.

II. SYSTEM DESCRIPTIONS

Let the fourth-order linear time-varying system C, and its series realization by third and first-order linear time-varying subsystems A and B be defined by the following differential equations

$$C: c_4(t)y^{iv}(t) + c_3(t)y'''(t) + c_2(t)y''(t) + c_1(t)y'(t) + c_0(t)y(t) = x(t), \tag{2.1}$$

where $x(t)$ and $y(t)$ are the input and output functions respectively, with $c_i(\bullet)$ as the coefficients of the time-varying system, which are piecewise continuous on $[t_0, \infty)$. Due to it's order of 4, $c_4(t) \neq 0$. Considering the decomposition of C as the cascade connection of first-order system A and third-order system B described as

$$A: a_1(t)y_A'(t) + a_0(t)y_A(t) = x_A(t), \tag{2.2}$$

$$B: b_3(t)y_B'''(t) + b_2(t)y_B''(t) + b_1(t)y_B'(t) + b_0(t)y_B(t) = x_B(t), \tag{2.3}$$

where $a_1(t) \neq 0$ and $b_3(t) \neq 0$. Also, a_i, b_i, x_A, $x_B \in P[t_0, \infty)$. Suppose that the series realization of C is the cascade connection of A and B is as in Fig. (1a) or (1b); the connections are abbreviated as AB or BA according to their sequence of connection.

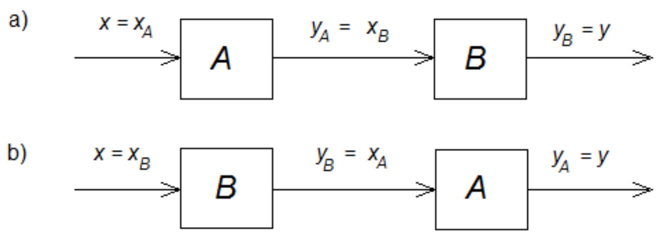

Fig. 1: Cascade connection of differential systems A and B

The propose is to find the subsystems A and B such that each one of the connections AB and BA are equivalent to the original system C in particular to find the conditions for the decomposition of the system C into it's subsystems A and B

and the coefficients of these subsystems. The found results are expressed by a theorem presented in the next section

III. Main Results

Theorem: *The necessary and sufficient conditions that a fourth-order linear time-varying system described by Eq. (2.1) into its cascade connected linear time-varying commutative pairs A and B of first and third-order respectively, are that*

i) There should be some constants k_3, k_2, k_1, k_0 such that the coefficients c_2, c_1 and c_0 can be expressible in terms of c_4 and c_3 as

$$c_2=\frac{1}{32}\Bigg[\frac{12c_3^2}{c_4}-\frac{12\sqrt{c_4}k_2^2}{k_3^{3/2}}+\frac{32\sqrt{c_4}k_1}{\sqrt{k_3}}+48c_3'-\frac{48c_3c_4'}{c_4}+\frac{35(c_4')^2}{c_4}-40c_4''\Bigg], \tag{3.1}$$

$$\begin{aligned}c_1=\frac{1}{512}\Bigg[&\frac{32c_3^3}{c_4^2}+\frac{64c_4^{1/4}k_2^3}{k_3^{9/4}}-\frac{96c_3k_2^2}{\sqrt{c_4}k_3^{3/2}}\\&-\frac{256c_4^{1/4}k_1k_2}{k_3^{5/4}}+\frac{256c_3k_1}{\sqrt{c_4}\sqrt{k_3}}+\frac{512c_4^{1/4}k_0}{k_3^{1/4}}\\&+\frac{384c_3c_3'}{c_4}-\frac{348c_3^2c_4'}{c_4^2}+\frac{36k_2^2c_4'}{\sqrt{c_4}k_3^{3/2}}+\frac{24c_3k_2c_4'}{c_4^{5/4}k_3^{3/4}}\\&-\frac{64k_1c_4'}{\sqrt{c_4}\sqrt{k_3}}+\frac{60k_2^2c_4'}{\sqrt{c_4}\sqrt{k_3}}-\frac{36c_3^2k_3c_4'}{c_4^2}\\&-\frac{24c_3k_2k_3^{1/4}c_4'}{c_4^{5/4}}-\frac{192k_1\sqrt{k_3}c_4'}{\sqrt{c_4}}-\frac{880c_3'c_4'}{c_4}\\&+\frac{1124c_3(c_4')^2}{c_4^2}-\frac{24k_2(c_4')^2}{c_4^{5/4}k_3^{3/4}}+\frac{24k_2k_3^{1/4}(c_4')^2}{c_4^{5/4}}\\&-\frac{144k_3c_3'c_4'}{c_4}+\frac{180c_3k_3(c_4')^2}{c_4^2}-\frac{955(c_4')^3}{c_4^2}\\&-\frac{165k_3(c_4')^3}{c_4^2}+512c_3''-\frac{832c_3c_4''}{c_4}+\frac{1592c_4'c_4''}{c_4}\\&+\frac{168k_3c_4'c_4''}{c_4}-640c_4'''\Bigg],\end{aligned} \tag{3.2}$$

$$\begin{aligned}c_0=\frac{1}{4096}\Bigg[&\frac{16c_3^4}{c_4^3}-\frac{48k_2^4}{k_3^3}+\frac{128c_3k_2^3}{c_4^{3/4}k_3^{9/4}}+\frac{256k_1k_2^2}{k_3^2}\\&-\frac{96c_3^2k_2^2}{c_4^{3/2}k_3^{3/2}}-\frac{512c_3k_1k_2}{c_4^{3/4}k_3^{5/4}}-\frac{384k_2^2c_3'}{\sqrt{c_4}k_3^{3/2}}+\frac{1024k_1c_3'}{\sqrt{c_4}\sqrt{k_3}}\\&-\frac{1024k_0k_2}{k_3}+\frac{256c_3^2k_1}{c_4^{3/2}\sqrt{k_3}}+\frac{1024c_3k_0}{c_4^{3/4}k_3^{1/4}}+\frac{384c_3^2c_3'}{c_4^2}\\&+\frac{768(c_3')^2}{c_4}-\frac{384c_3^3c_4'}{c_4^3}-\frac{192k_2^3c_4'}{c_4^{3/4}k_3^{9/4}}+\frac{576c_3k_2^2c_4'}{c_4^{3/2}k_3^{3/2}}\\&+\frac{768k_1k_2c_4'}{c_4^{3/4}k_3^{5/4}}-\frac{1536c_3k_1c_4'}{c_4^{3/2}\sqrt{k_3}}-\frac{648k_2^2(c_4')^2}{c_4^{3/2}k_3^{3/2}}\\&-\frac{1536k_0c_4'}{c_4^{3/4}k_3^{1/4}}-\frac{3584c_3c_3'c_4'}{c_4^2}+\frac{3096c_3^2(c_4')^2}{c_4^3}\\&+\frac{7264c_3'(c_4')^2}{c_4^2}-\frac{9504c_3(c_4')^3}{c_4^3}+\frac{8505(c_4')^4}{c_4^3}\\&+\frac{1728k_1(c_4')^2}{c_4^{3/2}\sqrt{k_3}}+\frac{1024c_3c_3''}{c_4}-\frac{3072c_4'c_3''}{c_4}\\&-\frac{1344c_3^2c_4''}{c_4^2}+\frac{576k_2^2c_4''}{\sqrt{c_4}k_3^{3/2}}-\frac{1536k_1c_4''}{\sqrt{c_4}\sqrt{k_3}}-\frac{4352c_3'c_4''}{c_4}\\&+\frac{10944c_3c_4'c_4''}{c_4^2}-\frac{17520(c_4')^2c_4''}{c_4^2}+\frac{4800(c_4'')^2}{c_4}\\&+1024c_3'''-\frac{2304c_3c_4'''}{c_4}+\frac{5760c_4'c_4'''}{c_4}-1536c_4^{iv}\Bigg].\end{aligned} \tag{3.3}$$

ii) Then, with the same constants k's, the coefficients of the decompositions A and B must be expressed in terms of c_4 and c_3 as

$$a_1=\left(\frac{c_4}{k_3}\right)^{1/4}, \tag{3.4}$$

$$a_0=\frac{1}{8}\left(\frac{2c_3-3k_3c_4'}{c_4^{3/4}k_3^{1/4}}-\frac{2k_2}{k_3}\right). \tag{3.5}$$

$$b_3=c_4^{3/4}k_3^{1/4}, \tag{3.6}$$

$$b_2=\frac{1}{8}\left[3\left(\frac{k_3}{c_4}\right)^{1/4}(2c_3-c_4')+2k_2\left(\frac{c_4}{k_3}\right)^{1/2}\right], \tag{3.7}$$

$$\begin{aligned}b_1=&\frac{1}{64}\left[\frac{c_4^{1/4}(-20k_2^2+64k_1k_3)}{k_3^{5/4}}\right]\\&+\frac{k_3^{1/4}[12c_3^2-60c_3c_4'+55(c_4')^2]}{64c_4^{5/4}}\\&+\frac{1}{8}\left[\frac{k_2(c_3-c_4')}{\sqrt{c_4}\sqrt{k_3}}+\frac{k_3^{1/4}(6c_3'-7c_4'')}{c_4^{1/4}}\right],\end{aligned} \tag{3.8}$$

$$b_0 = \frac{k_3^{1/4}}{512c_4^{9/4}}[8c_3^3 - 108c_3^2c_4' + 414c_3(c_4')^2 - 405(c_4')^3 + \frac{k_2[4c_3^2 - 24c_3c_4' + 27(c_4')^2]}{256c_4^{3/2}\sqrt{k_3}} + \frac{(-5k_2^2 + 16k_1k_3)(2c_3 - 3c_4')}{128c_4^{3/4}k_3^{5/4}} + \frac{k_3^{1/4}[6c_3(2c_3' - 5c_4'') + c_4'(-38c_3' + 75c_4'')]}{64c_4^{5/4}} + \frac{k_3^{1/4}(2c_3'' - 3c_4''')}{8c_4^{1/4}} + \frac{3k_2^3 - 16k_1k_2k_3 + 64k_0k_3^2}{64k_3^2}. \quad (3.9)$$

IV. Example

For the illustration of the decomposition conditions and the decomposition formulas presented in the previous section, we consider the following fourth-order linear time-varying system C:

$$t^4y^{iv}(t) + t^3y'''(t) + 2t^2y''(t) - \frac{23}{4}ty'(t) + \frac{111}{16}y(t) = x(t), \quad (4.1)$$

where the coefficients are

$$c_4 = t^4, c_3 = t^3, c_2 = 2t^2, c_1 = -\frac{23}{4}t, c_0 = \frac{111}{16}, \quad (4.2)$$

with the constants

$$k_3 = 1, k_2 = 1, k_1 = 1, k_0 = -2. \quad (4.3)$$

Condition (i) of Theorem are satisfied; that is c_2, c_1 and c_0 satisfy Eq. (3.1) and Eq. (3.2) respectively.

Condition (ii) of Theorem indicating the coefficients for decomposed subsystems A and B produces the following equations:

$$A:\ ty_A'(t) - \frac{3}{2}y_A(t) = x_A(t), \quad (4.4)$$

$$B: t^3y_B'''(t) - \frac{1}{2}t^2y_B''(t) + \frac{9}{4}ty_B'(t) - \frac{37}{8}y_B(t) = x_B(t). \quad (4.5)$$

Simulations are done with a sinusoid of amplitude 2, bias -3 and frequency 7 rad/sec, under a fixed step length of 0.01. By using ODE 45 [Dormand-Prince] as the solver, Simulink results are depicted in Fig. 2. All the decomposition conditions are satisfied and AB, BA and C give the same response as depicted in the figure ($AB = BA = C$). By replacing the input with a Saw-tooth of amplitude 2 and frequency 2 rad/sec, under zero initial condition and a fixed step length of 0.01. All the decomposition conditions are satisfied and $AB1, BA1$ and $C1$ give the same response as depicted in the figure ($AB1 = BA1 = C1$).

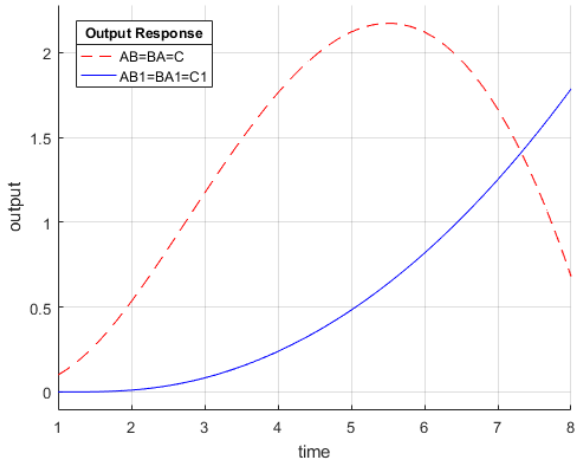

Fig. 2: Simulation results of Example

V. CONCLUSIONS

This presentation shows the results for the conditions that a fourth-order linear time-varying system C realized as a cascaded connection of linear first and third-order time-varying subsystems. In this context, explicate decomposition formulas for evaluating the subsystems A and B are presented. The results are verified to be correct by an example which is simulated by Simulink toolbox of MATLAB.

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