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Statistical Equi-Equal Convergence of Positive Linear Operators

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***Abstract* –** Many researchers have been interested in the concept of statistical convergence. Since statistical convergence is stronger than the classical convergence. Then, F. Móricz has introduced the statistical convergence of double sequences. Korovkin type approximation theorems have been investigated for sequences (or double sequences) of positive linear operators defined on different spaces via several new convergence methods. Also, it is known that, the concepts of statistical equal convergence and equi-statistical convergence are more general than the statistical uniform convergence. In this work a new type of statistical convergence is defined via using the notions of equi-statistical convergence and statistical equal convergence for double sequences to prove a Korovkin type approximation theorems. Show that the theorem is a non-trivial extension of some well-known Korovkin type approximation theorems which were demonstrated by earlier authors. We give an example in support of new definition and result presented in this work. Finally, we calculate the rate of statistical equi-equal convergence of double sequences of positive linear operators.

Keywords – Statistical equal convergence, double sequences, Korovkin theorem, equi statistical convergence.

I. Introduction

Firstly, we recall these convergence methods.

Let $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ be the two dimensional set of natural numbers and let $A \subset \mathbb{N}^2$. Also let

$$A_{mn} := \{(k, j) : k \leq m, j \leq n \, and \, (k, j) \in A\}$$

and suppose that the symbol $|A_{mn}|$ denotes the cardinality of A_{mn}. Then the natural double density of A is defined by

$$\delta_2(A) := P - \lim_{m,n} \frac{1}{mn} \left| \{(k, j) : k \leq m, j \leq n \, and \, (k, j) \in A\} \right|$$

provided that the limit exists. A given sequence (x_{mn}) is said to be statistically convergent to ℓ if, for every $\varepsilon > 0$, the following set:

$$K = K(\varepsilon) := \{(m,n) : |x_{mn} - \ell| \geq \varepsilon\}$$

has natural density zero [12]. This means that, for every $\varepsilon > 0$, we have

$$\delta_2(K) := P - \lim_{m,n} \frac{1}{mn} \left| \{k \leq m, j \leq n : |x_{mn} - \ell| \geq \varepsilon\} \right| = 0.$$

Then, we write $st_2 - \lim x_{mn} = \ell$. We know that, if every double sequences is convergent then it is statistically convergent to same limit, but the converse is not true.

Let g and g_{mn} belong to $C(B)$, which is the space of all continuous real valued functions on a compact subset B of the two dimensional real numbers and $\|g\|_{C(B)}$ denotes the usual supremum norm of g in $C(B)$. Throughout the paper, we use the following notation:

$$K_{mn}((v,t),\varepsilon) := \left| \{(m,n) : |g_{mn}(v,t) - g(v,t)| \geq \varepsilon\} \right|$$

$$D_{mn}(\varepsilon) := \left| \{(m,n) : \|g_{mn} - g\|_{C(B)} \geq \varepsilon\} \right|$$

where $(v,t) \in B, \varepsilon > 0, (m,n) \in \mathbb{N}^2$.

Definition 1.1. [11] (g_{mn}) is said to be pointwise statistically convergent to g on B if for every $\varepsilon > 0$ and for each $(x, y) \in B$,

$$P - \lim_{m,n \to \infty} \frac{K_{mn}((v,t),\varepsilon)}{mn} = 0.$$

Then, it is denoted by $g_{mn} \to g(st_2)$ on B.

Definition 1.2. [3] (g_{mn}) is said to be equi-statistically convergent to g on B if for every $\varepsilon > 0$,

$$P - \lim_{m,n \to \infty} \frac{K_{mn}((v,t),\varepsilon)}{mn} = 0,$$ uniformly with respect to (v,t),

which means that $P - \lim_{m,n \to \infty} \frac{\|K_{mn}(.,\varepsilon)\|}{mn} = 0$, for every $\varepsilon > 0$. Then, it is denoted by $g_{mn} \to g(equi - st_2)$ on B.

Definition 1.3. [11] (g_{mn}) is said to be statistically uniform convergent to g on B if for every $\varepsilon > 0$,

$$P - \lim_{m,n \to \infty} \frac{D_{mn}(\varepsilon)}{mn} = 0.$$

Then, it is denoted by $g_{mn} \rightrightarrows g(st_2)$ on B.

Recently, the definition of equal convergence for real functions have been introduced and have been improved this convergence by Császár and Laczkovich [4,5]. Later, the concepts of I and I^*-equal convergence with the help of ideals have been studied by Das, Dutta and Pal [6]. Finally, Okçu Şahin and Dirik have introduced the concept of the statistical equal convergence for double sequence [13]. Let's remember this definition.

Definition 1.4. [13] If there is a positive numbers double sequence (ε_{mn}) with $st_2-\lim \varepsilon_{mn}=0$ such that for any $(x,y)\in B$

$$P-\lim_{m,n\to\infty}\frac{U_{mn}\left(\left(v,t\right),\varepsilon\right)}{mn}=0$$

where $U_{mn}\left(\left(v,t\right),\varepsilon\right):=\left|\left\{\left(m,n\right):\left|g_{mn}\left(v,t\right)-g\left(v,t\right)\right|\geq\varepsilon_{mn}\right\}\right|$ then (g_{mn}) is said to be statistical equal convergent to g on B. In this case we write $g_{mn}\to g(eq-st_2)$ on B.

Now, we can introduce the concept of new our convergence for double sequences of functions.

Definition 1.5. If there is a positive numbers double sequence (ε_{mn}) with $st_2-\lim \varepsilon_{mn}=0$ such that,

$$P-\lim_{m,n\to\infty}\frac{V_{mn}\left(\left(v,t\right),\varepsilon_{mn}\right)}{mn}=0,$$ uniformly with respect to $\left(v,t\right)$,

where $V_{mn}\left(\left(v,t\right),\varepsilon_{mn}\right):=\left|\left\{\left(m,n\right):\left|g_{mn}\left(v,t\right)-g\left(v,t\right)\right|\geq\varepsilon_{mn}\right\}\right|$ then (g_{mn}) is said to be statistical equal convergent to g on B. In this case we write $g_{mn}\to g(equi-eq-st_2)$ on B.

Now we give an example which satisfies that statistical equi-equal convergence is stronger than statistical uniform convergence.

Example 1.1. Let $B=[0,1]\times[0,1]$, for each $\left(v,t\right)\in B$, $h\left(v,t\right)=0$ and (h_{mn}) is a double sequence of functions on B given by

$$h_{mn}\left(v,t\right)=\begin{cases} v^m t^n, & \text{if } m,n \text{ are squares},\\ 0, & \text{otherwise}.\end{cases} \tag{1.1}$$

Take (ε_{mn}) defined by

$$\varepsilon_{mn}=\begin{cases} 2m+n^2, & \text{if } m,n \text{ are squares},\\ \dfrac{1}{2m+3n}, & \text{otherwise}.\end{cases}$$

Then it is easy to see that $st_2-\lim \varepsilon_{mn}=0$. Also for any $\left(v,t\right)\in B$

$$\left\{\left(m,n\right):\left|h_{mn}\left(v,t\right)-h\left(v,t\right)\right|\geq\varepsilon_{mn}\right\}=\varnothing.$$

Therefore, we get $h_{mn}\to h(equi-eq-st_2)$ on B. But since $\sup_{(v,t)\in B}\left|h_{mn}\left(v,t\right)-h\left(v,t\right)\right|=1$ then (h_{mn}) is not statistical (or uniform) uniform convergence to the function $h=0$ on B.

II. Approximation of Operators

In 1961 Korovkin [15] studied the problem of the uniform convergence of $\left(S_n\left(g\right)\right)$ to a function g for a sequence (S_n) of positive linear operators defined on $C(B)$, by using the test function y^j, $j=0,1,2$ (also, see [2]). More recently, general versions of the Korovkin theorem were studied, in which a more general notion of convergence is used. Some Korovkin-type theorems in the setting of a statistical convergence were given by [1, 7, 8, 9, 10, 13].

In this section we apply the notion of statistical equi-equal convergence of a sequence of functions to prove a Korovkin type approximation theorem.

Let S be a linear operator from $C(B)$ into itself. Then, we state that S is positive linear operator on condition that $g\geq 0$ implies $S\left(g\right)\geq 0$. Also, we show the value of $S\left(g\right)$ at a point $(x,y)\in B$ by $S\left(g\left(u,t\right);x,y\right)$ or, briefly, $S\left(g;x,y\right)$.

Firstly, we recall the classical case of the Korovkin-type result as follows:

Theorem 2.1. [16] Let (S_{mn}) be a double sequence of positive linear operators acting $C(B)$ into $C\left(B\right)$. Then, we have

$$P-\lim_{m,n}\left\|S_{mn}\left(g\right)-g\right\|=0$$

if and only if

$$P-\lim_{m,n}\left\|S_{mn}\left(e_j\right)-e_j\right\|=0 \quad j=0,1,2,3,$$

where $e_0\left(v,t\right)=1$, $e_1\left(v,t\right)=v$, $e_2\left(v,t\right)=t$, $e_3\left(v,t\right)=v^2+t^2$.

Now we remember the following Korovkin-type approximation theorem by means of statistical uniform convergence.

Theorem 2.2. [7] Let (S_{mn}) be a double sequence of positive linear operators from $C(B)$ into $C\left(B\right)$. Then, we have

$$S_{mn}\left(g\right)\rightrightarrows g(st_2)$$

if and only if

$$S_{mn}\left(e_j\right)\rightrightarrows e_j(st_2),\quad j=0,1,2,3,$$

where $e_0\left(v,t\right)=1$, $e_1\left(v,t\right)=v$, $e_2\left(v,t\right)=t$, $e_3\left(v,t\right)=v^2+t^2$.

Now we have the following main result:

Theorem 2.3. Let (S_{mn}) be a double sequence of positive linear operators acting $C(B)$ into $C\left(B\right)$. Then, we have

(2.1) $S_{mn}(g) \to g(equi-eq-st_2)$

if and only if

(2.2) $S_{mn}(e_j) \to e_j(equi-eq-st_2), \quad j = 0,1,2,3,$

where $e_0(v,t) = 1$, $e_1(v,t) = v$, $e_2(v,t) = t$, $e_3(v,t) = v^2 + t^2$.

Proof: Condition (2.1) follows immediately from condition (2.2), since each of the functions $e_0(v,t) = 1$, $e_1(v,t) = v$, $e_2(v,t) = t$, $e_3(v,t) = v^2 + t^2$. belongs to $C(B)$. We prove the converse part. By the continuity of g on B, we can write

$$|g(u,x) - g(v,t)| \le 2\kappa .$$

Also, using continuity of g on B, we write that for $\forall \varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|g(u,x) - g(v,t)| < \varepsilon$$

whenever $|u-v| < \delta$ and $|x-t| < \delta$. Hence, putting $\psi(u,t) = (u-v)^2 + (x-t)^2$, we get

(2.3) $$|g(u,x) - g(v,t)| < \varepsilon + \frac{2\kappa}{\delta^2}\psi(u,t).$$

Since S_{mn} is positive linear operator, we write

$$\begin{aligned}|S_{mn}(g;v,t) - g(v,t)| &= |S_{mn}(g;v,t) - g(v,t)S_{mn}(e_0;v,t)\\ &\quad + g(v,t)S_{mn}(e_0;v,t) - g(v,t)|\\ &\le S_{mn}(|g(u,x) - g(v,t)|;v,t)\\ &\quad + \kappa|S_{mn}(e_0;v,t) - g(v,t)|\end{aligned} \quad (2.4)$$

Now we calculate the term of "$S_{mn}(|g(u,x) - g(v,t)|;v,t)$" to inequality (2.4)

$$\begin{aligned}S_{mn}(|g(u,x) - g(v,t)|;v,t) &\le S_{mn}\left(\varepsilon + \frac{2\kappa}{\delta^2}\psi(u,x);v,t\right)\\ &\le \varepsilon S_{mn}(e_0;v,t)\\ &\quad + \frac{2\kappa}{\delta^2}S_{mn}(\psi(u,x);v,t).\end{aligned} \quad (2.5)$$

Now we calculate the term of

"$S_{mn}(\psi(u,x);v,t)$"

to inequality (2.5),

$$\begin{aligned}S_{mn}(\psi(u,x);v,t) &= S_{mn}((u-v)^2 + (x-t)^2;v,t)\\ &\le |S_{mn}(e_3;v,t) - e_3(v,t)|\\ &\quad + 2\|e_2\|_{C(B)}|S_{mn}(e_2;v,t) - e_2(v,t)|\\ &\quad + 2\|e_2\|_{C(B)}|S_{mn}(e_1;v,t) - e_1(v,t)|\\ &\quad + \|e_3\|_{C(B)}|S_{mn}(e_0;v,t) - e_0(v,t)|.\end{aligned} \quad (2.6)$$

Using (2.5) and (2.6) in (2.4) we get

$$|S_{mn}(g;v,t) - g(v,t)| \le \varepsilon + \lambda\sum_{j=0}^{3}|S_{mn}(e_j;v,t) - e_j(v,t)|$$

where

$$\lambda := \max\left\{\varepsilon + \kappa + \frac{2\kappa}{\delta^2}\|e_3\|_{C(B)}, \frac{4\kappa}{\delta^2}\|e_1\|_{C(B)}, \frac{4\kappa}{\delta^2}\|e_2\|_{C(B)}, \frac{2\kappa}{\delta^2}\right\}.$$

Since ε is arbitrary, we can write

(2.7)
$$|S_{mn}(g;v,t) - g(v,t)| \le \lambda\sum_{j=0}^{3}|S_{mn}(e_j;v,t) - e_j(v,t)|.$$

$S_{mn}(e_j) \to e_j(equi-eq-st_2), \quad j = 0,1,2,3,$ there is a positive numbers double sequence $(\varepsilon_{mn}{}^{j})$ with $st_2 - \lim \varepsilon_{mn}{}^{j} = 0$ such that

$P - \lim\limits_{m,n\to\infty} \dfrac{V_{mn}((v,t),\varepsilon_{mn}{}^{j})}{mn} = 0$, uniformly with respect to (v,t),

where $V_{mn}{}^{j}((v,t),\varepsilon_{mn}) := \left|\left\{(m,n) : |S_{mn}(e_j;v,t) - e_j(v,t)| \ge \varepsilon_{mn}{}^{j}\right\}\right|$.

Then, for any $(v,t) \in B$

$$V_{mn}((v,t),\varepsilon_{mn}) := \left|\left\{(m,n) : |S_{mn}(g;v,t) - g(v,t)| \ge 4\lambda\varepsilon_{mn}\right\}\right|$$

where $\varepsilon_{mn} := \max\{\varepsilon_{mn}{}^{j} : j = 0,1,2,3\}$. It follows from (2.7) that

$$V_{mn}{}^{j}((v,t),\varepsilon_{mn}) \le \sum_{j=0}^{3} V_{mn}((v,t),\varepsilon_{mn}{}^{j})$$

and so

$$\frac{V_{mn}((v,t),\varepsilon_{mn})}{mn} \le \sum_{j=0}^{3}\frac{V_{mn}{}^{j}((v,t),\varepsilon_{mn}{}^{j})}{mn}$$

Then using the hypothesis (2.2) we get

$$S_{mn}(g) \to g(equi-eq-st_2).$$

This completes the proof of the theorem.

Now, we present an example in support of above result.

Example 2.1. Let $B = [0,1]\times[0,1]$, and consider the classical double Bernstein polynomials on $C(B)$;

$$B_{mn}(g;v,t) = \sum_{k=0}^{m}\sum_{s=0}^{n} g\left(\frac{k}{m},\frac{s}{n}\right)\binom{m}{k}\binom{n}{s}v^k t^s(1-v)^{m-k}(1-t)^{n-s}.$$

Now, we introduce positive linear operators as follows:

(2.8) $$S_{mn}(g;v,t)=(1+h_{mn}(v,t))B_{mn}(g;v,t),$$
$(v,t)\in B, g\in C(B)$

where $h_{mn}(v,t)$ given by (1.1) in Example 1.1. Then, we see that

$$S_{mn}(e_0;v,t)=(1+h_{mn}(v,t))e_0(v,t),$$
$$S_{mn}(e_1;v,t)=(1+h_{mn}(v,t))e_1(v,t),$$
$$S_{mn}(e_2;v,t)=(1+h_{mn}(v,t))e_2(v,t),$$
$$S_{mn}(e_3;v,t)=(1+h_{mn}(v,t))\times\left\{e_3(v,t)+\frac{v-v^2}{m}+\frac{t-t^2}{n}\right\}$$

Since
$h_{mn}\to h=0(equi-eq-st_2)$, on $j=0,1,2,3$
we obtain that
$S_{mn}(e_j)\to e_j(equi-eq-st_2)$ for each $j=0,1,2,3$.
So, by Theorem 2.3, we have, for all $g\in C(B)$,
$S_{mn}(g)\to g(equi-eq-st_2)$.
Furthermore, since $\sup_{(v,t)\in B}|h_{mn}(v,t)-h(v,t)|=1$, we can say that the results given in Theorem 2.1. and Theorem 2.2., respectively, do not hold true for our operators defined by (2.8).

III. Rate of the Statistical equi equal convergence

In this section, we study the corresponding rates of our convergence for double sequences with the help of modulus of continuity.

Now, we remind that the modulus of continuity of a function $g\in C(B)$ is defined by

$$\omega_2(g;\delta)=\sup_{\substack{\sqrt{(u-v)^2+(x-t)^2}\le\delta,\\(u,x),(v,t)\in B}}|g(u,x)-g(v,t)| \quad (\delta>0).$$

Also, it is known that for any $\delta>0$ and each $(u,x),(v,t)\in B$

$$|g(u,x)-g(v,t)|\le\omega_2(g;\delta)\left(\frac{\sqrt{(u-v)^2+(x-t)^2}}{\delta^2}+1\right).$$

Then we have the following result:

Theorem 3.1. Let (S_{mn}) be a double sequence of positive linear operators acting $C(B)$ into $C(B)$. Assume that the following conditions hold:

(a) $S_{mn}(e_0)\to e_0(equi-eq-st_2)$ on B,

(b) $\omega_2(g;\delta)\to 0(equi-eq-st_2)$ on B

$\delta_{mn}(v,t)=\sqrt{S_{mn}(\Psi_{(v,t)};v,t)}$ with

$$\Psi_{(v,t)}(u,x)=(u-v)^2+(x-t)^2.$$

Then we have, for all $g\in C(B)$,

(c) $S_{mn}(g)\to g(equi-eq-st_2)$.

Proof. Let $g\in C(B)$ and $(v,t)\in B$. It is known that

$$|S_{mn}(g;v,t)-g(v,t)|\le\kappa|S_{mn}(e_0;v,t)-e_0(v,t)|+\left\{S_{mn}(e_0;v,t)+\frac{S_{mn}(\Psi_{(v,t)};v,t)}{\delta^2}\right\}\omega_2(f;\delta)$$

where $\kappa:=\|f\|_{C(B)}$. If we choose

$$\delta:=\delta_{mn}(v,t):=\sqrt{S_{mn}(\Psi_{(v,t)};v,t)},$$

this yield that

(3.1)
$$|S_{mn}(g;v,t)-g(v,t)|\le\kappa|S_{mn}(e_0;v,t)-e_0(v,t)|+\omega_2(f;\delta_{mn})|S_{mn}(e_0;v,t)-e_0(v,t)|+2\omega_2(f;\delta_{mn}).$$

$S_{mn}(e_0)\to e_0(equi-eq-st_2)$, there is a positive numbers double sequence $(\varepsilon_{mn,0})$ with $st_2-\lim\varepsilon_{mn,0}=0$ such that

$P-\lim_{m,n\to\infty}\frac{V_{mn}^0((v,t),\varepsilon_{mn,0})}{mn}=0$, uniformly with respect to (v,t),

where $V_{mn}^0((v,t),\varepsilon_{mn,0}):=\left|\{(m,n):|S_{mn}(e_0;v,t)-e_0(v,t)|\ge\varepsilon_{mn,0}\}\right|$.

$\omega_2(g;\delta)\to 0(equi-eq-st_2)$ there is a positive numbers double sequence $(\varepsilon_{mn,1})$ with $st_2-\lim\varepsilon_{mn,1}=0$ such that

$P-\lim_{m,n\to\infty}\frac{V_{mn}^1((v,t),\varepsilon_{mn,1})}{mn}=0$, uniformly with respect to (v,t),

where $V_{mn}^1((v,t),\varepsilon_{mn,1}):=\left|\{(m,n):\omega_2(g;\delta)\ge\varepsilon_{mn,1}\}\right|$. Then, for any $(v,t)\in B$

$$V_{mn}((v,t),\varepsilon_{mn}):=\left|\{(m,n):|S_{mn}(g;v,t)-g(v,t)|\ge\varepsilon_{mn}\}\right|$$

where $\varepsilon_{mn,2}:=\max\{\varepsilon_{mn,j}:j=0,1\}$,

$$\varepsilon_{mn}:=\varepsilon_{mn,2}^2+(\kappa+2)\varepsilon_{mn,2}.$$

It follows from (3.1) that

$$V_{mn}((v,t),\varepsilon_{mn})\le\sum_{j=0}^{1}V_{mn}^j((v,t),\varepsilon_{mn,j})$$

Then using the hypothesis (a) and (b), proof is completed.

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