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Statistical Equal Convergence on Weighted Spaces

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Abstract – The Korovkin theory has effective role in approximation theory. This theory is connected with the approximation to continuous functions by means of positive linear operators. Many mathematicians have investigated the Korovkin-type theorems by for a sequence of positive linear operators defined on different spaces by using various types of convergence. Firstly, A.D. Gadjiev has proved the weighted Korovkin type theorems, (Math. Zamet., 20 (1976) 781-786 (in Russian)). Later, these theorems are studied by many authors by means of different convergence methods. Recently, The definition of equal convergence for real functions was introduced by Császár and Laczkovich and they improved their investigations on this convergence. Later Das et. al. introduced the ideas of I and I^*-equal convergence with the help of ideals by extending the equal convergence (Mat. Vesnik, vol:66, 2 (2014),165-177). In our work, we introduce a new type of statistical convergence on weighted spaces by using the notions of the equal convergence. We study its use in the Korovkin-type approximation theory. Then, we construct an example such that our new approximation result works but its classical and statistical cases do not work.

Keywords – Statistical equal convergence, Double sequences, Korovkin theorem, Equi-statistical convergence.

I. INTRODUCTION

Now we remind the concepts of weight functions and weight spaces. The function $\rho:\mathbb{R}\to\mathbb{R}$ is called a weight function if it is continuous on $\mathbb{R}$, $\lim_{|y|\to\infty}\rho(y)=\infty$ and for all $y\in\mathbb{R}$, $\rho(y)\geq 1$. Then the corresponding space of real valued functions f defined on $\mathbb{R}$ and satisfying $|f(y)|\leq M_f\rho(y)$ (for all $y\in\mathbb{R}$) is called weighted space and denoted by B_ρ, where M_f is a constant depending on the function f. The weighted subspace C_ρ of B_ρ is given by

$$C_\rho=\{f\in B_\rho : f \text{ is continuous on } \mathbb{R}\}.$$

Then C_ρ and B_ρ are Banach spaces with the norm (see [1])

$$\|g\|_\rho=\sup_{y\in\mathbb{R}}\frac{|f(y)|}{\rho(y)}.$$

Let ρ_1 and ρ_2 be two weight functions satisfying below conditions. Also assume that

$$\lim_{|y|\to\infty}\frac{\rho_1(y)}{\rho_2(y)}=0. \qquad (1.1)$$

If T is a positive linear operator from C_{ρ_1} into B_{ρ_2}, then the operator norm $\|T\|_{C_{\rho_1}\to B_{\rho_2}}$ is given by

$$\|T\|_{C_{\rho_1}\to B_{\rho_2}}=\sup_{\|f\|_{\rho_1}=0}\|Tf\|_{\rho_2}.$$

Throughout this paper, we use the test functions $F_j, j=0,1,2$ defined by

$$F_0(y)=\frac{\rho_1(y)}{1+y^2}, F_1(y)=\frac{y\rho_1(y)}{1+y^2}, F_2(y)=\frac{y^2\rho_1(y)}{1+y^2}.$$

Császár and Laczkovich was introduced the definition of equal convergence for real functions on a compact subset I of the real numbers [2,3]. Later, Das, Dutta and Pal introduced the ideas of I and I^*-equal with the help of ideals by extending the equal convergence [4].

Let's remember this definition. Let $\mathbb{N}$ be the set of natural numbers and let $A\subset\mathbb{N}$. Also let

$$A_n:=\{k\leq n : k\in A\}$$

and suppose that the symbol $|A_n|$ denotes the cardinality of A_n. Then the natural double density of A is defined by

$$\delta(A):=\lim_n\frac{1}{n}|\{k\leq n : k\in A\}|$$

provided that the limit exists. A given sequence (y_n) is said to be statistically convergent to l if, for every $\varepsilon>0$, the following set:

$$K=K(\varepsilon):=\{n : |y_n-l|\geq\varepsilon\}$$

has natural density zero [9], i.e., for $\forall\varepsilon>0$, we have

$$\delta(K):=\lim_n\frac{1}{n}|\{k\leq n : |y_n-l|\geq\varepsilon\}|=0.$$

In this case, we show $st-\lim y_n=l$. It is known that, every convergent sequence is statistically convergent to same limit, but the converse is not true.

Let's remember this definition.

Definition 1.1. [4] If there is a positive numbers sequence (ε_n) with $st-\lim\varepsilon_n=0$ such that for any $y\in I$

$$\lim_{n\to\infty}\frac{\left|\left\{n:\left|f_n(y)-f(y)\right|\geq\varepsilon_n\right\}\right|}{n}=0,$$

then (f_n) is said to be statistical equal convergent to f on I. In this case we write $f_n \to f(eq-st)$ on I.

Let f and f_n belong to C_{ρ_1}.

Definition 1.2. (f_n) is said to be statistical equal convergent to f on C_{ρ_1}, if there is a positive numbers sequence (ε_n) with $st-\lim\varepsilon_n=0$ such that for any $y\in I$

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{n:\left|\frac{f_n(y)-f(y)}{\rho_1(y)}\right|\geq\varepsilon_n\right\}\right|=0,$$

Then, we write $f_n \overset{\rho_1}{\to} f(eq-st)$ on I

Now we give an example.

Example 1.1. Let, for each $y\in\mathbb{R}$, $h(y)=0$ and (h_n) is a sequence of functions on $\mathbb{R}$ given by

$$h_n(y)=\begin{cases}\dfrac{1}{1+ny^2}, & if\ n\ is\ square,\\ 0, & otherwise.\end{cases} \tag{1.2}$$

Let $\rho_1(y)=1+y^2$. Then every $n\in\mathbb{N}$, $h_n\in C_{\rho_1}$. Take (ε_n) defined by

$$\varepsilon_n=\begin{cases}n^2+1, & if\ n\ is\ square,\\ \dfrac{1}{2n}, & otherwise.\end{cases}$$

Then it is easy to see that $st-\lim\varepsilon_n=0$. Also for any $y\in\mathbb{R}$

$$\left\{n:\left|\frac{h_n(y)-h(y)}{\rho_1(y)}\right|\geq\varepsilon_n\right\}=\varnothing.$$

Therefore, we get $h_n \overset{\rho_1}{\to} h(eq-st)$ on $\mathbb{R}$. But since $\sup_{y\in\mathbb{R}}\left|h_n(y)-h(y)\right|=1$ then (h_n) is not statistical and classical uniform convergence to the function $h=0$ on $\mathbb{R}$.

II. APPROXIMATION BY MEANS OF STATISTICAL EQUAL CONVERGENCE

In this section we prove a Korovkin type approximation theorem by means of the concept of statistical equal convergence.

Let T be a linear operator from C_{ρ_1} into B_{ρ_2}. If $h\geq 0$ implies $T(h)\geq 0$, then we say that T is positive linear operator. Also, we denote the value of $T(h)$ at a point $y\in\mathbb{R}$ by $T(h(u);y)$ or, briefly, $T(h;y)$.

We need the following Lemmas to prove our main theorem.

Lemma 2.1. Let (T_n) be a double sequence of positive linear operators from C_{ρ_1} into B_{ρ_2}. If

$$T_n(F_j)\overset{\rho_1}{\to}F_j(eq-st),\ j=0,1,2, \tag{2.1}$$

then, for any $b>0$ and for any $|y|\leq b$, we have

$$T_n(F)\overset{\rho_2}{\to}F(eq-st),\ for\ all\ F\in C_{\rho_1}.$$

Proof. Let $F\in C_{\rho_1}$ and $|y|\leq b$. Since F is continuous on $\mathbb{R}$, we write that for every $\varepsilon>0$, there exists a number $\delta>0$ such that $\left|F(u)-F(y)\right|<\varepsilon$ whenever $|u-y|<\delta$. If $|u-y|\geq\delta$, then we obtain

$$\begin{aligned}\left|F(u)-F(y)\right| &\leq 2M_F\rho_1(y)F_0(u)\left(1+y^2\right)\\ &\leq 4M_F\rho_1(y)F_0(u)(u-y)^2\\ &\quad\times\left(1+\frac{1+y^2}{(u-y)^2}\right)\\ &\leq S_{\rho_1}(y)F_0(u)(u-y)^2\end{aligned}$$

where $S_{\rho_1}(y):=4M_F\rho_1(y)\left(1+\dfrac{1+y^2}{\delta^2}\right)$. For all $u\in\mathbb{R}$ and $|y|\leq b$, we have

$$\left|F(u)-F(y)\right|\leq\varepsilon+S_{\rho_1}(y)F_0(u)(u-y)^2. \tag{2.2}$$

Then, we write

$$\begin{aligned}&\left|T_n(F;y)-F(y)\right|\\ &\leq T_n\left(\left|F(u)-F(y)\right|;y\right)\\ &+\left|F(y)\right|\left|T_n(1;y)-1\right|\\ &\leq T_n\left(\varepsilon+K_{\rho_1}(y)F_0(u)(u-y)^2;y\right)\\ &+\left|F(y)\right|\left|T_n(1;y)-1\right|\\ &=\varepsilon T_n(1;y)+K_{\rho_1}(y)T_n\left(F_0(u)(u-y)^2;y\right)\\ &+\left|F(y)\right|\left|T_n(1;y)-1\right|.\end{aligned}$$

Hence, all $y\in\mathbb{R}$ with $|y|\leq b$, we have

$$\begin{aligned}\left|T_n(F;y)-F(y)\right| &\leq\varepsilon M_1\frac{T_n(1;y)}{\rho_1(y)}\\ &+M_2T_n\left(F_0(u)(u-y)^2;y\right)\\ &+M_3\left|T_n(1;y)-1\right|\end{aligned} \tag{2.3}$$

where

$$M_1 := M_1(b) := \sup_{|y|\le b} \rho_1(y),$$
$$M_2 := M_2(b) := \sup_{|y|\le b} S_{\rho_1}(y),$$
$$M_3 := M_3(b) := \sup_{|y|\le b} |F(y)|.$$

For any $y \in \mathbb{R}$ with $|y| \le b$ and $b > 0$

(2.4)
$$T_n\left(F_0(u)\left[(u-y)^2\right];y\right) \le \left\{|T_n(F_2;y)-F_2(y)| + 2|y||T_n(F_1;y)-F_1(y)| + (y^2)|T_n(F_0;y)-F_0(y)|\right\}$$
$$\le M_4\left\{\frac{|T_n(F_0;y)-F_0(y)|}{\rho_1(y)} + \frac{|T_n(F_1;y)-F_1(y)|}{\rho_1(y)} + \frac{|T_n(F_2;y)-F_2(y)|}{\rho_1(y)}\right\}$$

where $M_4 := M_4(B) = \max\left\{\sup_{|y|\le b}\rho_1(y), 2\sup_{|y|\le b}|y|\rho_1(y), \sup_{|x|\le b}(y^2)\rho_1(y)\right\}$. Since $F_0 \in C_{\rho_1}$ and

$$F_0(y)|T_n(1;y)-1| \le T_n\left(|F_0(u)-F_0(y)|;y\right) + |T_n(F_0;y)-F_0(y)|,$$

It follows from (2.3), that

$$|T_n(1;y)-1| \le \frac{1}{F_0(y)}\{\varepsilon\, T_n(1;y) + |T_n(F_0;y)-F_0(y)| + S_{\rho_1}(y)\, T_n\left(F_0(y)(u-y)^2;y\right)\}.$$

Then, we have, for any $y \in \mathbb{R}$ with $|y| \le b$ and $b > 0$ and for all $n \in \mathbb{N}$, that

(2.5)
$$|T_n(1;y)-1| \le M_5\left\{\frac{|T_n(F_0;y)-F_0(y)|}{\rho_1(y)} + \varepsilon\frac{T_n(1;y)}{\rho_1(y)}\right\} + M_6 T_n\left(F_0(u)(u-y)^2;y\right)$$

where

$$M_5 := M_5(b) := \sup_{|y|\le b}\frac{\rho_1(y)}{F_0(y)}$$

and

$$M_6 := M_6(b) := \sup_{|y|\le b}\frac{S_{\rho_1}(y)}{F_0(y)}.$$

For all $n \in \mathbb{N}$,

$$\sup_{|y|\le b}\frac{T_n(1;y)}{\rho_1(y)} \le \sup_{|y|\le b}\frac{T_n(\rho_1;y)}{\rho_1(y)} \le \sup_{|y|\le b}\frac{|T_n(\rho_1;y)-\rho_1(y)|}{\rho_1(y)} + 1$$
$$\le \sup_{|y|\le b}\frac{|T_n(F_2;y)-F_2(y)|}{\rho_1(x)} + \sup_{|y|\le b}\frac{|T_n(F_0;y)-F_0(y)|}{\rho_1(y)} + 1.$$

Since

(2.6)
$$\sup_{|y|\le b}\frac{T_n(1;y)}{\rho_1(y)} < \infty$$

and considering the (2.3), (2.4) and (2.5), we obtain

$$|T_n(F;y)-F(y)| \le M\left\{\varepsilon\frac{T_n(1;y)}{\rho_1(y)} + \frac{|T_n(F_0;y)-F_0(y)|}{\rho_1(y)} + \frac{|T_n(F_1;y)-F_1(y)|}{\rho_1(y)} + \frac{|T_n(F_2;y)-F_2(y)|}{\rho_1(y)}\right\}$$

where

$$M := \max\left\{M_1 + M_3M_5, M_4(M_2 + M_3H_6) + M_3M_5\right\}.$$

By using (1.6), taking $H = \max\left\{M\sup_{|y|\le b}\frac{T_n(1;y)}{\rho_1(y)}, M\right\}$ and since ε arbitrary, we get

(2.7)
$$\frac{|T_n(F;y)-F(y)|}{\rho_2(y)} \le H\left\{\sum_{t=0}^{2}\frac{|T_n(F_t;y)-F_n(y)|}{\rho_1(y)}\right\}$$

for all $n \in \mathbb{N}$ for some $H > 0$ independent of y. Since $T_n(F_j) \overset{\rho_1}{\rightarrow} F_j(eq-st), j = 0,1,2$ on $\mathbb{R}$, there are positive number sequence $(\varepsilon_{n,t})$ with $st - \lim_n \varepsilon_{n,t} = 0$ such that, for any $y \in \mathbb{R}$,

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{n : \left|\frac{T_n(F_t;y)-F_t(y)}{\rho_1(y)}\right| \ge \varepsilon_{n,t}\right\}\right| = 0.$$

For any $y \in \mathbb{R}$ with $|y| \le b$, from (2.7), we have

$$\left\{n : \left|\frac{T_n(F;y)-F(y)}{\rho_2(y)}\right| \ge 3\varepsilon_n\right\} \subset \left\{n : \left|\frac{T_n(F_t;y)-F_t(y)}{\rho_1(y)}\right| \ge \varepsilon_{n,t}\right\}$$

where $\varepsilon_n = \max\{\varepsilon_{n,t} : t = 0,1,2\}$. Then using the hypothesis (2.1), we get, for any $y \in \mathbb{R}$ with $|y| \le b$, for all $F \in C_{\rho_1}$,

$$T_n(F) \overset{\rho_1}{\rightarrow} F(eq-st)$$

The proof of lemma is complete.

Theorem 1.1. Let ρ_1 and ρ_2 be as in Lemma 1.1. Let (T_n) is a sequence of positive linear operators acting C_{ρ_1} into B_{ρ_2}. Then for all $F \in C_{\rho_1}$,

(2.8) $$T_n(F) \overset{\rho_2}{\rightarrow} F(eq-st),$$

on condition that

(2.9) $$T_n(F_j) \overset{\rho_1}{\rightarrow} F_j(eq-st), j = 0,1,2$$

Proof. We can show that the hypothesis (2.9) implies $T_n(F_t; y) - F_t(y) \in B_{\rho_1}$ and therefore $T_n(F_t; y) \in B_{\rho_1}$ for $t = 0,1,2$. Since $\rho_1 = F_0 + F_2$, we obtain $T_n(\rho_1) \in B_{\rho_1}$ for each n. If $F \in C_{\rho_1}$ then, we can write $T_n(F) \in B_{\rho_1}$. Besides, we get, for $H_1 > 0$,

$$\|T_n\|_{C_{\rho_1} \to B_{\rho_1}} = \|T_n(\rho_1)\|_{\rho_1} = \sup_{y \in \mathbb{R}} \frac{T_n(\rho_1; y)}{\rho_1(y)} \le H_1 < \infty.$$

Hence, we obtain for a given $F \in C_{\rho_1}$, that

(2.10) $$\|T_n(F)\|_{\rho_1} \le \|T_n\|_{C_{\rho_1} \to B_{\rho_1}} \|F\|_{\rho_1} \le H_1 \|F\|_{\rho_1}.$$

For a given $\varepsilon > 0$, choose an $b_0 > 0$ such that $\frac{\rho_1(y)}{\rho_2(y)} \le \varepsilon$ for every $|y| \ge b_0$. This is possible by (1.1). Using the fact, we obtain for $F \in C_{\rho_1}$ and for any $y \in \mathbb{R}$ with $|y| \ge b_0$, also by (2.10),

$$\begin{aligned}\frac{|T_n(F;y) - F(y)|}{\rho_2(y)} &= \frac{|T_n(F;y) - F(y)|}{\rho_2(y)} \frac{\rho_1(y)}{\rho_1(y)} \\ &\le \varepsilon \frac{|T_n(F;y) - F(y)|}{\rho_1(y)} \\ &\le \varepsilon \left(\|T_n(F)\|_{\rho_1} + \|F\|_{\rho_1} \right) \\ &\le \varepsilon \|F\|_{\rho_1} (H_1 + 1)\end{aligned}$$

then, for $F \in C_{\rho_1}$ and for any $y \in \mathbb{R}$ with $|y| \ge b_0$

(2.11) $$T_n(F) \overset{\rho_2}{\rightarrow} F(eq-st).$$

Also, using the Lemma 1.1, for any $b > 0$ and for all $F \in C_{\rho_1}$, we have for any $y \in \mathbb{R}$ with $|y| \le b$,,

(2.12) $$T_n(F) \overset{\rho_2}{\rightarrow} F(eq-st),$$

then, we get from (2.11) and (2.12),

$$T_n(F) \overset{\rho_2}{\rightarrow} F(eq-st) \text{ on } \mathbb{R},$$

the proof is completed.

Now we get the following statistical result:

Theorem 1.2. [5] Let ρ_1 and ρ_2 are weight functions satisfying (1.1). Suppose that (T_n) is a sequence of positive linear operators from C_{ρ_1} into B_{ρ_2}. Then, for all $F \in C_{\rho_1}$,

$$st - \lim \|T_n(F) - F\|_{\rho_2} = 0$$

if

$$st - \lim \|T_n(F_t) - F_t\|_{\rho_1} = 0, \; t = 0,1,2.$$

Theorem 1.3. [6,7] Let ρ_1 and ρ_2 are weight functions satisfying (1.1). Suppose that (T_n) is a sequence of positive linear operators from C_{ρ_1} into B_{ρ_2}. Then, for all $F \in C_{\rho_1}$,

$$\lim \|T_n(F) - F\|_{\rho_2} = 0$$

if

$$\lim \|T_n(F_t) - F_t\|_{\rho_1} = 0, \; t = 0,1,2.$$

III. APPLICATION

In this part, we give an example of a sequence of positive linear operators that satisfies the conditions of Theorem 1.1 but do not satisfy the conditions of Theorem 1.

Example 3.1. Consider the following linear positive operators given in [8] which is defined by:

(3.1) $$L_n(F;x)y := \sum_{v=0}^{\infty} F\left(\frac{v}{\beta_n}\right) K_{n,v}(y) \frac{(-\alpha_n)^v}{v!}.$$

Here (α_n) and (β_n) be the real number sequences satisfying the followings:

(i) $\lim_{n\to\infty} \beta_n = \infty$, (ii) $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0$, (iii) $\lim_{n\to\infty} n \frac{\alpha_n}{\beta_n} = 1$,

and $K_{n,v}(x)$ is the functions satisfy the following conditions:

a) For any natural $n, v = 0,1,2,...$ and for any $y \in [0,\infty)$

$$(-1)^v K_{n,v}(y) \ge 0$$

b) For any $y \in [0,\infty)$

$$\sum_{v=0}^{\infty} K_{n,v}(y) \frac{(-\alpha_n)^v}{v!} = 1,$$

c) $K_{n,v}(y) = -nyK_{n+m,v-1}(y)$ for any $y \in [0,\infty)$ where $n+m$ is a natural numbers and m is a constant independent of v.

Then, using the operators $L_n(F;y)$, introduce the following positive linear operators:

(3.2) $$T_n(F;y) = (1 + h_n(y)) L_n(F;y),$$

where (h_n) given by (1.2) in Example 1.1. Also taking, $\rho_1(x)=1+y^2$ and $\rho_2(y)$ arbitrary such as satisfying the condition (1.1) holds. Then, we obtain the test functions $F_0(y)=1,\ F_1(y)=y,$ and $F_2(y)=y^2$. We claim that

(3.3) $T_n(F_j)\overset{\rho_1}{\rightarrow}F_j(eq-st), j=0,1,2$ on $\mathbb{R}$.

We can show that

$$T_n(F_0;y)=(1+h_n(y))F_0(y),$$

$$T_n(F_1;y)=(1+h_n(y))n\frac{\alpha_n}{\beta_n}F_1(y),$$

$$T_n(F_2;y)=(1+h_n(y))\left(n(n+m)\frac{\alpha_n^2}{\beta_n^2}y^2+n\frac{\alpha_n}{\beta_n^2}y\right).$$

Then,

$$|T_n(F_0;y)-F_0(y)|=|(1+h_n(y))-1|=h_n(y),$$

and $h_n\overset{\rho_1}{\rightarrow}h(eq-st)$ on $\mathbb{R}$, then,

$$T_n(F_0)\overset{\rho_1}{\rightarrow}F_0(eq-st) \text{ on } \mathbb{R},$$

so, (3.3) holds true for $t=0$. It is obvious that

$$|T_n(F_1;y)-F_1(y)|=\left|(1+h_n(y))n\frac{\alpha_n}{\beta_n}y-y\right|$$

$$=|y|\left|(1+h_n(y))n\frac{\alpha_n}{\beta_n}-1\right|$$

$$\leq|y|\left(\left|n\frac{\alpha_n}{\beta_n}-1\right|+\left|h_n(y)n\frac{\alpha_n}{\beta_n}\right|\right),$$

then, by means of (iii) and $h_n\overset{\rho_1}{\rightarrow}h(eq-st)$ on $\mathbb{R}$,

$$\lim_{n\to\infty}\left(n\frac{\alpha_n}{\beta_n}-1\right)=0,$$

$$n\frac{\alpha_n}{\beta_n}h_n\overset{\rho_1}{\rightarrow}0(eq-st) \text{ on } \mathbb{R}.$$

Since $\sup_{y\in[0,\infty)}\frac{|y|}{1+y^2}<\infty$, we get

$$T_n(F_1)\overset{\rho_1}{\rightarrow}F_1(eq-st) \text{ on } \mathbb{R},$$

then, (3.3) holds true for $t=1$. Finally, since

$$|T_n(F_2;y)-F_2(y)|$$

$$=\left|(1+h_n(y))\left[n(n+m)\frac{\alpha_n^2}{\beta_n^2}y^2+n\frac{\alpha_n}{\beta_n^2}y\right]-y^2\right|$$

$$\leq|y^2|\left(\left|n(n+m)\frac{\alpha_n^2}{\beta_n^2}-1\right|+\left|h_n(y)n(n+m)\frac{\alpha_n^2}{\beta_n^2}\right|\right)$$

$$+|y|\left|(1+h_n(y))n\frac{\alpha_n}{\beta_n^2}\right|$$

because of (iii) and and $h_n\overset{\rho_1}{\rightarrow}h(eq-st)$ on $\mathbb{R}$, we can show that

(3.4) $$\lim_n\left(n(n+m)\frac{\alpha_n^2}{\beta_n^2}-1\right)=0,$$

$$n(n+m)\frac{\alpha_n^2}{\beta_n^2}h_n\overset{\rho_1}{\rightarrow}0(eq-st) \text{ on } \mathbb{R},$$

and since $\lim_n\frac{1}{\beta_n}=0$, from (i) and also using (iii), we get

(3.5) $$\lim_n n\frac{\alpha_n}{\beta_n^2}=0,$$

$$n(n+m)\frac{\alpha_n}{\beta_n^2}h_n\overset{\rho_1}{\rightarrow}0(eq-st) \text{ on } \mathbb{R}.$$

Since $\sup_{y\in[0,\infty)}\frac{y^2}{1+y^2}<\infty$, $\sup_{y\in[0,\infty)}\frac{|y|}{1+y^2}<\infty$ and from (3.4), (3.5), we obtain

$$T_n(F_3)\overset{\rho_1}{\rightarrow}F_3(eq-st) \text{ on } \mathbb{R}.$$

Hence, our claim (3.3) holds true for each $r=0,1,2$. (T_n) satisfies all hypothesis of Theorem 1.1 and we see that, for all $F\in C_{\rho_1}$,

$$T_n(F)\overset{\rho_2}{\rightarrow}F(eq-st), \text{ on } \mathbb{R}.$$

However, since

$$\frac{|T_n(F_0;y)-F_0(y)|}{\rho_1(y)}=\frac{|(1+h_n(y))F_0(y)-F_0(y)|}{\rho_1(y)}=\frac{h_n(y)}{\rho_1(y)},$$

then Theorem 1.2, Theorem 1.3 do not work for the sequence (T_n).