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A Bound for the Derivative of Positive Real Functions and Corresponding Circuits

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Abstract *– –* In this paper, driving point impedance functions, $Z(s) = A + c_1(s-b) + c_2(s-b)^2 + \ldots$, which are frequently used in electrical engineering, have been considered for boundary analysis of the Schwarz lemma. Accordingly, considering the s_1, s_2, …, s_n points in the right half plane which are different than $s = b$, Schwarz lemma has been obtained for positive real functions. In addition, a result of the Rogosinski's lemma has been used to prove the new inequalities and the derivative of the driving point impedance function has been evaluated from below by considering Taylor expansion coefficients c_1 and c_2. For all presented inequalities, sharpness analysis has been carried out and extremal functions corresponding to different driving point impedance functions have been obtained. It is possible to say that simple circuits can be synthesized using the obtained transfer functions.

Keywords – Schwarz lemma, Analytic function, Taylor expansion, Driving point impedance function, Rogosinski lemma

I. INTRODUCTION

Positive real functions are used as driving point impedance functions in electrical engineering. The conditions for positive real functions can be listed as follows [1]:

1-) $Z(s)$ is analytic and single valued in $\Re s \geq 0$ except possibly for poles on the axis of imaginaries,

2-) $\overline{Z(s)} = Z(\bar{s})$

3-) $\Re Z(s) \geq 0$, in $\Re s \geq 0$.

In this study, a lemma and a theorem are presented by using the derivative of the DPI function, $Z(s)$. Accordingly, new inequalities are obtained for the modulus of derivative of $Z(s)$ evaluated at $s = 0$, that is $|Z'(0)|$. Derivative of DPI functions are frequently used in electrical engineering [2-8]. By sharpness analysis of the obtained inequalities, novel driving point impedance functions can be determined. Here, generic circuit schematic is presented using the DPI functions derived in the presented theorem. In addition, magnitude and phase graphics are given for different parameters of the DPI function.

The rest of the manuscript is organized as follows: The preliminary considerations for the theorem will be presented in Section II. In Section III, main results are presented and at the end, in Section IV, conclusions are given.

II. PRELIMINARY CONSIDERATIONS

The classical Schwarz lemma says that an analytic function f from the unit disk $E = \{z : |z| < 1\}$ into itself with f(0) = 0 must map each smaller disk $\{z : |z| < r < 1\}$ into itself. Further, $|f'(0)| \leq 1$ and $|f'(0)| = 1$ if and only if f is a rotation of E. This is a very powerful tool in complex analysis [9,10]. A sharpened version of this is Rogosinski's lemma [11], which say that for all $z \in E$

$$|f(z) - d_1| \leq r_1,$$

where

$$d_1 = \frac{zf'(0)\left(1-|z|^2\right)}{1-|z|^2|f'(0)|^2},\ r_1 = \frac{|z|^2\left(1-|f'(0)|^2\right)}{1-|z|^2|f'(0)|^2}.$$

Consider the function

$$g(z) = \frac{Z(s)-A}{Z(s)+A}, z = \frac{s-b}{s+b},$$

where $Z(s) = A + c_1(s-b) + c_2(s-b)^2 + \ldots$ and A, b are positive real numbers. These mappings are also used in classical circuit theory [12].

Consider the product

$$\upsilon(z) = \prod_{i=1}^{n} \frac{z - z_i}{1 - \overline{z_i} z},$$

where $z_1, z_2, \ldots, z_n \in E$.
The auxiliary function

$$\phi(z)=\frac{g(z)}{\vartheta(z)}=\frac{Z\left(b\frac{1+z}{1-z}\right)-A}{Z\left(b\frac{1+z}{1-z}\right)+A}\frac{1}{\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}.$$

Here, $\phi(z)$ is an analytic function in E, $\phi(0)=0$ and $|\phi(z)|<1$ for $|z|<1$. From the definition of $\phi(z)$, we have

$$\phi(z)=\frac{c_1\frac{2bz}{1-z}+c_2\left(\frac{2bz}{1-z}\right)^2+...}{2A+c_1\frac{2bz}{1-z}+c_2\left(\frac{2bz}{1-z}\right)^2+...}\frac{1}{\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}.$$

Therefore, we take

$$\frac{\phi(z)}{z}=\frac{c_1\frac{2b}{1-z}+c_2\left(\frac{2b}{1-z}\right)^2z+...}{2A+c_1\frac{2bz}{1-z}+c_2\left(\frac{2bz}{1-z}\right)^2+...}\frac{1}{\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}.$$

Therefore, from Schwarz lemma, we obtain

$$|Z'(b)|\leq\frac{A}{b}\prod_{i=1}^{n}|z_i|.$$

The result is sharp and the extremal function is

$$Z(s)=A\frac{1+\frac{s-b}{s+b}\prod_{i=1}^{n}\frac{\frac{s-b}{s+b}-\frac{s_i-b}{s_i+b}}{1-\frac{\overline{s_i-b}}{s_i+b}\frac{s-b}{s+b}}}{1-\frac{s-b}{s+b}\prod_{i=1}^{n}\frac{\frac{s-b}{s+b}-\frac{s_i-b}{s_i+b}}{1-\frac{\overline{s_i-b}}{s_i+b}\frac{s-b}{s+b}}}.$$

We thus obtain the following lemma.

Lemma. Let $Z(s)=A+c_1(s-b)+c_2(s-b)^2+...$ be a Positive Real Function. Suppose that $s_i, s_2, ..., s_n$ are points in the right half plane that are different from $s=1$ with $Z(s_i)=A$. Then

$$|Z'(b)|\leq\frac{A}{b}\prod_{i=1}^{n}|z_i|$$

(1.1)

The inequality (1.1) is sharp, with equality for the function

$$Z(s)=A\frac{1+\frac{s-b}{s+b}\prod_{i=1}^{n}\frac{\frac{s-b}{s+b}-\frac{s_i-b}{s_i+b}}{1-\frac{\overline{s_i-b}}{s_i+b}\frac{s-b}{s+b}}}{1-\frac{s-b}{s+b}\prod_{i=1}^{n}\frac{\frac{s-b}{s+b}-\frac{s_i-b}{s_i+b}}{1-\frac{\overline{s_i-b}}{s_i+b}\frac{s-b}{s+b}}},$$

where $s_i, s_2, ..., s_n$ are positive real numbers and $\overline{s_i-b}=s_i-b$. Then, the equality can be rewritten as follows:

$$Z(s)=A\frac{1+\frac{s-b}{s+b}\prod_{i=1}^{n}\frac{s-s_i}{s+s_i}}{1-\frac{s-b}{s+b}\prod_{i=1}^{n}\frac{s-s_i}{s+s_i}}$$

For different n values, it is possible to design different electrical circuit schematics using Cauer 1 realization.

For n = *1*, the Z(s) function is given $Z(s)=A\frac{s^2+bs_1}{s(s_1+b)}$ and corresponding circuit is given as shown in Fig.1.

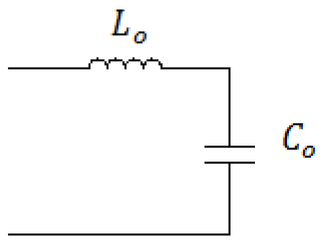

Figure 1. Corresponding circuit for *n = 1* where $L_o=\frac{A}{s_1+b}H$ ve $C_o=\frac{bs_1}{(s_1+b)}F$.

For simplicity, assume that A=1, s_1=1 and b=1 for $n=1$. Then, the DPI function can be rewritten as $Z(s)=\frac{s^2+1}{2s}=\frac{1}{2}s+\frac{1}{2s}$. Accordingly, the values of inductor and capacitor in Fig. 1 is given as $L_o=\frac{1}{2}H$ ve $C_o=2F$, respectively. The frequency characteristics of the circuit are shown in Fig. 2. As it can be seen from Fig. 2, the resonance frequency is w=1 rad/s.

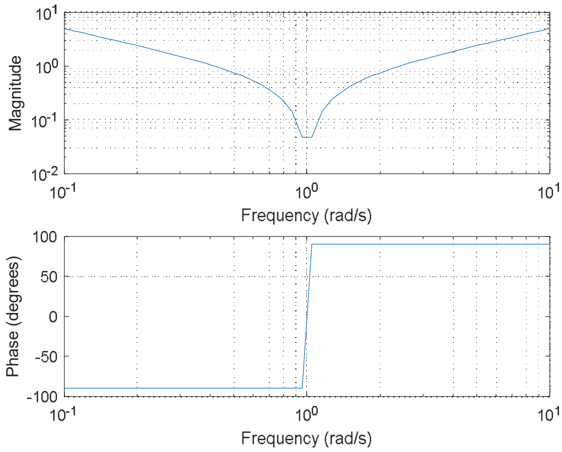

Figure 2. Frequency characteristic graphic of $Z(s) = \frac{1}{2}s + \frac{1}{2s}$

For odd values of n, the generalized circuit model is given in Fig.3.

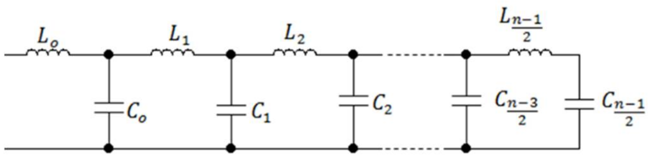

Figure 3. Generalized circuit model of the impedance function given in Lemma for odd values of n.

For even values of n, generalized circuit model is given as in Fig. 4.

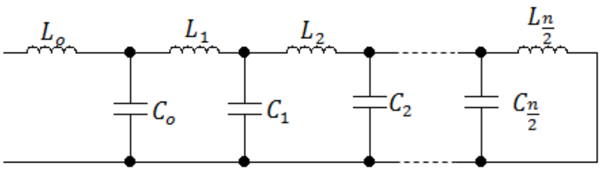

Figure 4. Generalized circuit model of the impedance function given in Lemma for even values of n.

An elementary consequence of the Schwarz lemma is that if $f(z)$ extends continuously to some boundary point c, then $|f(c)| = 1$ and, if $f(z)$ is differentiable at c, $|f'(c)| \geq 1$. The boundary Schwarz lemma is also a fundamental tool in the study of the geometric properties of functions of several complex variables [13-18].

III-MAIN RESULTS

In this section, a theorem which gives an inequality for the derivative of DPI function, $Z(s)$, is presented with proof. In the presented theorem, assuming that $Z(0)=0$ and $Z(s)$ is a positive real function which is given as $Z(s) = A + c_1(s-b) + c_2(s-b)^2 + \ldots$, a lower bound has been obtained for $|Z'(0)|$. The extremal function has been obtained by carrying out the sharpness analysis and corresponding circuit has been designed using this function as shown in Fig. 5.

Theorem. Let $Z(s) = A + c_1(s-b) + c_2(s-b)^2 + \ldots$ be a positive real function that is also analytic at the point $s=0$ of the imaginary axis with $Z(0)=0$. Then

$$|Z'(0)| \geq \frac{A}{b}\left(1 + \frac{2\left(A - b|c_1|\right)^2}{A^2 - b^2|c_1|^2 + b\left|\left(c_1 + 2bc_2\right)A - bc_1^2\right|}\right). \quad (2.1)$$

The result (2.1) is sharp for the function given by

$$Z(s) = 2bA\frac{s}{s^2+b^2}.$$

Proof. Let

$$g(z) = \frac{Z(s)-A}{Z(s)+A}, z = \frac{s-b}{s+b}.$$

Let us consider the functions

$$h(z) = \frac{g(z)}{z}, \omega(z) = \frac{h(z)-h(0)}{1-\overline{h(0)}h(z)}.$$

$\omega(z)$ is analytic in E, $\omega(0)=0$ and $|\omega(z)|<1$ for $|z|<1$ and

$$\omega'(0) = \frac{h'(0)}{1-|h'(0)|^2} = \frac{g''(0)}{2\left(1-|g'(0)|^2\right)}.$$

From Rogosinski's lemma and [14], we have

$$|g(z) - d_2| \leq r_2,$$

where

$$d_2 = \frac{z|g'(0)|\left(1-m^2\right)}{1-m^2|g'(0)|^2}, \quad r_2 = \frac{m|z|\left(1-|g'(0)|^2\right)}{1-m^2|g'(0)|^2},$$

$$m = |z|\frac{|z|+|\omega'(0)|}{1+|z||\omega'(0)|}.$$

Therefore, we take

$$\left|\frac{g(z)-1}{z-1}\right| \geq \frac{1-|d_2|-r_2}{1-|z|}$$

$$= \frac{1-\dfrac{|z||g'(0)|\left(1-m^2\right)}{1-m^2|g'(0)|^2}-\dfrac{m|z|\left(1-|g'(0)|^2\right)}{1-m^2|g'(0)|^2}}{1-|z|}$$

$$= \frac{1+m|g'(0)|-|z||g'(0)|-m|z|}{\left(1-|z|\right)\left(1+m|g'(0)|\right)}.$$

If we replace the value of m with the last inequality, we get

$$\left|\frac{g(z)-1}{z-1}\right| \geq \frac{1+|z|+|z|^2+|z||\omega'(0)|\left(1-|z|\right)}{1+|z||\omega'(0)|+|z|^2|g'(0)|+|z||g'(0)||\omega'(0)|}$$

$$-\frac{|z||g'(0)|\left(1-|z|\right)+|z||g'(0)||\omega'(0)|\left(1-|z|\right)}{1+|z||\omega'(0)|+|z|^2|g'(0)|+|z||g'(0)||\omega'(0)|}.$$

Passing to the limit in the last inequality yields

$$|g'(-1)| \geq \frac{3+|\omega'(0)|-|g'(0)|+|g'(0)||\omega'(0)|}{1+|\omega'(0)|+|g'(0)|+|g'(0)||\omega'(0)|}$$

$$= \frac{3+|\omega'(0)|-|g'(0)|+|g'(0)||\omega'(0)|}{\left(1+|\omega'(0)|\right)\left(1+|g'(0)|\right)}$$

$$= 1+\frac{4\left(1-|g'(0)|\right)^2}{2\left(1-|g'(0)|^2\right)+|g''(0)|}.$$

Since

$$|g'(-1)| = \frac{b}{A}|Z'(0)|,\quad |g'(0)| = \frac{b}{A}|c_1|$$

$$|g''(0)| = \frac{2b}{A^2}\left|\left(c_1+2bc_2\right)A-bc_1^2\right|,$$

we obtain

$$\frac{b}{A}|Z'(0)| \geq 1+\frac{4\left(1-\dfrac{b}{A}|c_1|\right)^2}{2\left(1-\left(\dfrac{b}{A}|c_1|\right)^2\right)+\dfrac{2b}{A^2}\left|\left(c_1+2bc_2\right)A-bc_1^2\right|},$$

$$|Z'(0)| \geq \frac{A}{b}\left(1+\frac{4\left(A-b|c_1|\right)^2}{2\left(A^2-b^2|c_1|^2\right)+2b\left|\left(c_1+2bc_2\right)A-bc_1^2\right|}\right)$$

and

$$|Z'(0)| \geq \frac{A}{b}\left(1+\frac{2\left(A-b|c_1|\right)^2}{A^2-b^2|c_1|^2+b\left|\left(c_1+2bc_2\right)A-bc_1^2\right|}\right).$$

The equality case of the inequality is obtained as

$$Z(s) = 2bA\frac{s}{s^2+b^2}$$

As in the proposed lemma, realization of the impedance function given above can be carried out using Cauier 1 realization. Accordingly, obtained circuit structure is shown in Fig. 5 where $L_o = \frac{2A}{b}$ and $C_o = \frac{1}{2bA}$.

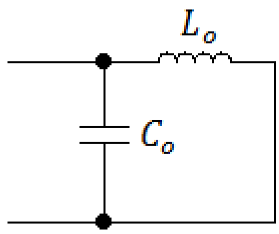

Figure 5. Corresponding circuit for Z(s) obtained in Theorem where $L_o = \frac{2A}{b}H, C_o = \frac{1}{2bA}$.

It is possible to obtain frequency characteristics of the circuit given in Fig. 5 by assigning arbitrary values for the parameters A and b. As an example, assume that A=1 and b=2. Then, the DPI function is rewritten as $(s) = 4\frac{s}{s^2+4} = \frac{1}{\frac{1}{4}s+\frac{1}{s}}$. Accordingly, the values of inductor and capacitor in Fig. 5 are obtained as 1 H and $\frac{1}{4}F$, respectively. The resonance frequency is determined as *w=2* rad/s. The obtained frequency and phase characteristic graphics are given in Fig. 6. As it can be seen from the figures, spike in the magnitude and phase change are located at *w=2* rad/s.

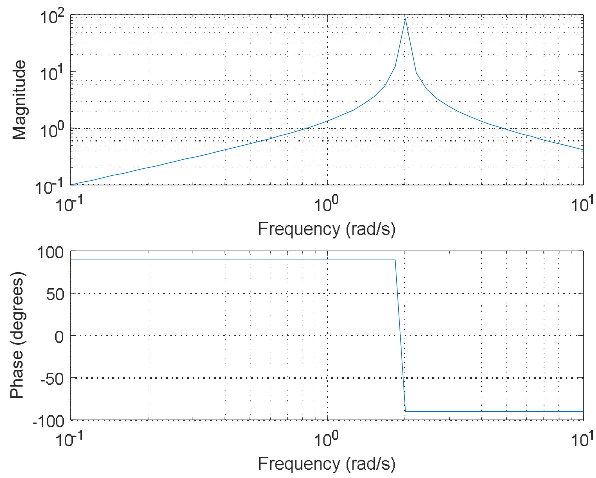

Figure 6. Frequency characteristic graphic of $Z(s) = \frac{1}{\frac{1}{4}s+\frac{1}{s}}$.

III. CONCLUSION

In this study, a theorem with a lemma is presented using the modulus of the derivative of the DPI functions. Accordingly, two inequalities are obtained in the proposed lemma and theorem. The equality case of these inequalities is determined by the sharpness analysis. As these equalities correspond to

DPI functions in electrical engineering, their frequency characteristics are also investigated. A generic equation which depends on *n* parameter is obtained in lemma. It is observed that simple LC circuits can be designed for even and odd values of n parameter. A tank circuit is obtained in theorem by using the obtained DPI function. It can be inferred from the results that different filter structures containing inductors and capacitors can be synthesized using the proposed lemma and theorem.

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