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ON SOME PROPERTIES OF LORENTZ-SOBOLEV SPACES WITH VARIABLE EXPONENT

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Abstract – In recent years there has been an increasing interest in the study of various mathematical problems with variable exponent Lebesgue spaces. There are also a lot of published papers in these spaces. Spaces of weakly differentiable functions, so called Sobolev spaces, play an important role in modern Analysis. The theory of variable exponent Sobolev spaces is useful theoretical tool to study the variable exponent problems, such as solutions of elliptic and parabolic partial differentiable equations, calculus of variations, nonlinear analysis, capacity theory and compact embeddings. Moreover, several authors studied some continuous embeddings from Sobolev spaces to Lorentz spaces. These kinds of embedding results are very interesting and valuable in analysis, and there are many applications of them in various fields. In this paper we define variable exponent Lorentz-Sobolev spaces and prove the boundedness of maximal function in these spaces. Also we will show that there is a continuous embedding between variable exponent Lorentz-Sobolev and Lorentz spaces under some conditions.

Keywords *– Variable exponent Lorentz and Sobolev spaces, Maximal function, embedding*

I. Introduction

It is well-known that the Sobolev spaces $W^{k,p}(\mathbb{R}^n)$ is continuously embedded into $L^{\frac{np}{n-kp}}(\mathbb{R}^n)$ for integer $k \geq 1, n \geq 1$ and real number $p \geq 1$, such that $n \geq kp$. This kinds of embedding results are very important in analysis and there are many important applications of them in various fields. The embedding of Sobolev spaces into Lorentz spaces was first considered by Alvino ([2], [3]), Brezis and Waigner [4], O'neil [21]. In the work of Helein [16], using the following embedding

$$W^{1,1}_{loc}(\mathbb{R}^2) \hookrightarrow L^{2,1}_{loc}(\mathbb{R}^2),$$

he proved the regularity of weakly harmonic mapping from surfaces to Riemann manifolds.

In recent years there has been an increasing interest in the study of various mathematical problems with variable exponent Lebesgue spaces (see [5],[6],[9],[11],[8], [13]). The theory of variable exponent Sobolev spaces is an important theoretical tool to study the variable exponent problems. The Sobolev embeddings theorems in the variable exponent Sobolev space $W^{1,p(.)}(\Omega)$ have been studied by many authors. Also Diening [10] proved the optimal Sobolev embedding $W^{k,p(.)}(\mathbb{R}^n) \hookrightarrow L^{p^*(.)}(\mathbb{R}^n)$ with $\frac{1}{p^*(.)} = \frac{1}{p(.)} - \frac{k}{n}$ and $W^{1,p(.)}(\Omega) \hookrightarrow L^{p^*(.)}(\Omega)$ for $k = 1$, where $\Omega \subset \mathbb{R}^n$ be an open, bounded set with Lipschitz boundary. In 1994 M. Jiang [17] proved that $W^{s,p}(\mathbb{R}^n) \hookrightarrow L^{\frac{np}{n-sp},p}(\mathbb{R}^n)$ for $n \geq 1$, $s > 0$, $p \geq 1$ and $n > sp$. In this paper, we discuss that the embedding $W^{1,p(.),q(.)}(\Omega) \hookrightarrow L^{\frac{np(.)}{n-p(.)},q(.)}(\Omega)$ is satisfied under which conditions for $1 < p(.) \leq p^+ < n$ and $1 < q(.) \leq q^+ < \infty$. We also prove that the boundedness of the maximal operator in $W^{1,p(.),q(.)}(\Omega)$ under the some conditions.

II. PRELIMINARIES

Throughout this paper all sets and functions are Lebesgue measurable. The Lebesgue measure and the characteristic function of a subset $A \subset \mathbb{R}^n$ will be denoted by $\mu(A) = |A|$ and χ_A respectively. The space $L^1_{loc}(\mathbb{R}^n)$ consists of all (classes of) measurable functions f on $\mathbb{R}^n$ such that $f\chi_K \in L^1(\mathbb{R}^n)$ for any compact subset $K \subset \mathbb{R}^n$. It is a topological vector space with the family of seminorms $f \mapsto \|f\chi_K\|_1$. A Banach function space (shortly BF-space) on $\mathbb{R}^n$ is a Banach space $(B, \|.\|_B)$ of measurable functions which is continously embedded into $L^1_{loc}(\mathbb{R}^n)$, i.e. for any compact subset $K \subset \mathbb{R}^n$ there exists some constant $C_K > 0$ such that $\|f\chi_K\|_1 \leq C_K\|f\|_B$ for all $f \in B$. We denote it by $B \hookrightarrow L^1_{loc}(\mathbb{R}^n)$. The class $C_0^\infty(\mathbb{R}^n)$ is defined as set of infinitely differentiable functions with compact support in $\mathbb{R}^n$.

Definition 1. Let (G, Σ, μ) be a measure space and let f be a measurable function on G. For each $y > 0$ let

$$\lambda_f(y) = \mu\{x \in G: |f(x)| > y\}.$$

The function λ_f is called the distribution function of f. The rearrangement of f is defined by

$$f^*(t) = inf\{y > 0: \lambda_f(y) \leq t\}$$
$$= sup\{y > 0: \lambda_f(y) > t\}, t > 0,$$

with the conventions $inf\emptyset = \infty$ and $sup\emptyset = 0$. Also, the average function of f is defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(x)dx, f^*(t) \leq f^{**}(t).$$

It is easy to see that λ_f, f^* and f^{**} are nonincreasing and right continuous on $(0, \infty)$.

Definition 2. For a measurable function $p: \mathbb{R}^n \to [1, \infty)$ (called a variable exponent on $\mathbb{R}^n$), we put

$$p^- = inf_{x\in\mathbb{R}^n} p(x), \quad p^+ = sup_{x\in\mathbb{R}^n} p(x).$$

The variable exponent Lebesgue spaces $L^{p(.)}(\mathbb{R}^n)$ consist of all measurable functions f such that $\rho_{p(.)}(\gamma f) < \infty$ for some $\gamma > 0$, equipped with the Luxemburg norm

$$\|f\|_{p(.)} = inf\left\{\gamma > 0: \rho_{p(.)}\left(\frac{f}{\gamma}\right) \leq 1\right\},$$

where

$$\rho_{p(.)}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx.$$

If $p^+ < \infty$, then $f \in L^{p(.)}(\mathbb{R}^n)$ iff $\rho_{p(.)}(f) < \infty$. The space $\left(L^{p(.)}(\mathbb{R}^n), \|.\|_{p(.)}\right)$ is a Banach space. If $p(.) = p$ is a constant function, then the norm $\|.\|_{p(.)}$ coincides with the usual Lebesgue norm $\|.\|_p$ [18]. In this paper we assume that $p^+ < \infty$.

In the one-dimensional case $n = 1$ we deal with the interval $[0, l]$, $0 < l \leq \infty$ and the standart Lebesgue measure. Let

$$p^- = inf_{x\in[0,l]} p(x), \qquad p^+ = sup_{[0,l]} p(x).$$

We will use the notation

$$\mathcal{P}_a = \{p: a < p^- \leq p^+ < \infty\}, a \in \mathbb{R}$$

and will be interested in the special cases of the classes $\mathcal{P}_a$ with $a = 0,1$.

By $\mathcal{P}([0, l])$ we denote the class of functions $p \in L^\infty([0, l])$ such that there exist the limits

$$p(0) = \lim_{x\to 0} p(x) \text{ and } p(\infty) = \lim_{x\to\infty} p(x),$$

the conditions at infinity being only needed in the case $l = \infty$. We also denote

$$\mathbb{P}_a([0, l]) = \mathcal{P}([0, l]) \cap \mathcal{P}_a([0, l]).$$

Definition 3. Let Ω be an open set in $\mathbb{R}^n$. We denote $l = \mu(\Omega)$ for brevity. Let $p(.), q(.) \in \mathcal{P}_0([0, l])$. By $L^{p(.),q(.)}(\Omega)$ we denote the space of functions f on Ω such that $t^{\frac{1}{p(t)}-\frac{1}{q(t)}} f^*(t) \in L^{q(.)}([0, l])$, i.e.,

$$\rho_{p(.),q(.)}(f) := \int_0^l t^{\frac{q(t)}{p(t)}-1} |f^*(t)|^{q(t)} dt < \infty$$

for $l < \infty$, and we use the notation

$$\|f\|_{p(.),q(.)} = inf\left\{\gamma > 0: \rho_{p(.),q(.)}\left(\frac{f}{\gamma}\right) \leq 1\right\}$$
$$= \left\|t^{\frac{1}{p(t)}-\frac{1}{q(t)}} f^*(t)\right\|_{L^{q(.)}([0,l])}.$$

The following properties were proved by several authors [14] and [19]:

1. Let $p(.), q(.) \in \mathcal{P}_1([0, l])$ and $p(0), p(\infty) > 1$. Then $L^{p(.),q(.)}(\Omega)$ is a Banach function spaces. Hence we have $L^{p(.),q(.)}(\Omega) \hookrightarrow L^1_{loc}(\Omega)$ and the Sobolev spaces of $L^{p(.),q(.)}(\Omega)$ is well defined.

2. Let $p(.), q(.) \in \mathcal{P}_1$. Then the dual space $\left(L^{p(.),q(.)}(\Omega)\right)^*$ is $L^{r(.),t(.)}(\Omega)$, where $\frac{1}{p(.)} + \frac{1}{r(.)} = 1$ and $\frac{1}{q(.)} + \frac{1}{t(.)} = 1$.

Definition 4. For $x \in \Omega$ and $r > 0$ we denote an open ball with center x and radius r by $B(x, r)$. For $f \in L^1_{loc}(\Omega)$, we denote the (centered) Hardy-Littlewood maximal operator Mf of f by

$$Mf(x) = sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(x)|\, dx,$$

where the supremum is taken over all balls $B(x, r)$.

The boundedness of the maximal operator in $L^{p(.)}$ was first proved by L. Diening [9] over bounded domains, under the assumption that $p(.)$ is locally log-Hölder continuous, that is,

$$|p(x) - p(y)| \leq \frac{C}{-ln|x-y|}, x, y \in \Omega, |x - y| \leq 1/2. \qquad (1)$$

In the case of unbounded sets in Ω, it is also supposed that the log-Hölder decay condition (at infinity)

$$|p(x) - p_\infty| \leq \frac{C}{log(e+|x|)}$$

is satisfied for some $p_\infty > 1$, $C > 0$ and all $x \in \Omega$. The locally log-Hölder condition (1) is no more needed for the boundedness of the maximal operator in $L^{p(.),q(.)}(\Omega)$ [14].

3. Let $p(.), q(.) \in \mathcal{P}_1([0, l])$ and $p(0), p(\infty) > 1$. Then the maximal operator is bounded in $L^{p(.),q(.)}(\Omega)$ [14].

Definition 5. Let $\varphi: \Omega \to \mathbb{R}$ be a nonnegative, radial, decreasing function belonging to $C_0^\infty(\Omega)$ and having the properties:

(i) $\varphi(x) = 0$ if $|x| \geq 1$, and

(ii) $\int_\Omega \varphi(x)\, dx = 1$.

For $k > 0$, if the function $\varphi_k(x) = k^{-n}\varphi\left(\frac{x}{k}\right)$ is nonnegative, belongs to $C_0^\infty(\Omega)$, and satisfies

(i) $\varphi_k(x) = 0$ if $|x| \geq k$, and

(ii) $\int_\Omega \varphi_k(x)\, dx = 1$,

where Ω is an open set in $\mathbb{R}^n$, then φ_k is called a mollifier.

The following Proposition was proved in [12].

Proposition 6. Let φ_k be a mollifier and $f \in L^1_{loc}(\Omega)$. Then

$$sup_{k>0} |(\varphi_k * f)(x)| \leq Mf(x).$$

Proposition 7. Let $p(.), q(.) \in \mathbb{P}_1([0, \infty])$, $q(\infty) \leq p(\infty)$ and $p(0) = q(0)$. If $f \in L^{p(.),q(.)}(\Omega)$, then $\varphi_k * f \to f$ in $L^{p(.),q(.)}(\Omega)$ as $k \to 0^+$.

Proof. Let $f \in L^{p(.),q(.)}(\Omega)$ and $0 < \varepsilon < 1$ be given. By Proposition 6, we have

$$\|\varphi_k * f\|_{p(.),q(.)} \leq \|Mf\|_{p(.),q(.)}$$
$$\leq C\|f\|_{p(.),q(.)},$$

and $\varphi_k * f \in L^{p(.),q(.)}(\Omega)$ for all $k > 0$ by Theorem 3.12 in [14]. It is also known that $C_c(\Omega)$ is dense in $L^{p(.),q(.)}(\Omega)$. Hence there is a function $g \in C_c(\Omega)$ such that

$$\|f - g\|_{p(.),q(.)} \leq {}^{\varepsilon}/_{2(c+1)}, \qquad (2)$$

and $\varphi_k * g \in C_0^\infty(\Omega)$ for all $k > 0$ by Theorem 2.29 in [1]. Using the technique in 3.1 Lemma in [19] we obtain

$$\rho_{p(.),q(.)}(f) \leq \|f\|_{q^+}^{q^+} + \|f\|_{q^-}^{q^-}.$$

So we can write

$$\rho_{p(.),q(.)}(g - \varphi_k * g) \leq \|g - \varphi_k * g\|_{q^+}^{q^+} + \|g - \varphi_k * g\|_{q^-}^{q^-}$$
$$< \frac{1}{2}\left(\frac{\varepsilon}{2}\right)^{q^+} + \frac{1}{2}\left(\frac{\varepsilon}{2}\right)^{q^-}$$
$$< \frac{1}{2}\frac{\varepsilon}{2} + \frac{1}{2}\frac{\varepsilon}{2} = \frac{\varepsilon}{2}$$

as $k \to 0^+$ by Theorem 5.4 in [7]. Since $\rho_{p(.),q(.)}(g - \varphi_k * g) < 1$, then we find

$$\rho_{p(.),q(.)}\left((g - \varphi_k * g)\rho_{p(.),q(.)}(g - \varphi_k * g)^{-\frac{1}{q^+}}\right) \leq 1.$$

By the definition of $\|.\|_{p(.),q(.)}$ we have

$$\|g - \varphi_k * g\|_{p(.),q(.)} \leq \rho_{p(.),q(.)}(g - \varphi_k * g)^{-\frac{1}{q^+}}$$
$$< \left(\frac{\varepsilon}{2}\right)^{q^+} < \frac{\varepsilon}{2}. \qquad (3)$$

Finally using the inequalities (2) and (3), we obtain

$$\|f - \varphi_k * f\|_{p(.),q(.)} < \varepsilon.$$

The proof is complete.

Corollary 8. $C_0^\infty(\Omega)$ is dense in $L^{p(.),q(.)}(\Omega)$.

III. Results

Definition 9. Variable exponent Lorentz-Sobolev space $W^{k,p(.),q(.)}(\Omega)$ is defined by

$$W^{k,p(.),q(.)}(\Omega)=\{f\in L^{p(.),q(.)}(\Omega): D^{\alpha}f\epsilon L^{p(.),q(.)}(\Omega), 0\leq |\alpha|\leq k, k\epsilon\mathbb{N}\}$$

equipped with the norm

$$\|f\|_{k,p(.),q(.)}=\sum_{0\leq|\alpha|\leq k}\|D^{\alpha}f\|_{p(.),q(.)},$$

where $\alpha\in\mathbb{N}_0^n$ is a multi-index, $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n$ and $D^{\alpha}=\frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1}\ldots\partial_{x_n}^{\alpha_n}}$. Note that $L^{p(.),q(.)}(\Omega)\hookrightarrow L^1_{loc}(\Omega)$, so that $L^{p(.),q(.)}(\Omega)$ functions are in $D'(\Omega)$. Thus $D^{\alpha}f$ makes sense as a distribution. It can be show that $W^{k,p(.),q(.)}(\Omega)$ is a reflexive Banach space.

The space $W^{1,p(.),q(.)}(\Omega)$ is defined by

$$W^{1,p(.),q(.)}(\Omega)=\{f\in L^{p(.),q(.)}(\Omega): |\nabla f|\epsilon L^{p(.),q(.)}(\Omega)\}.$$

The norm $\|f\|_{1,p(.),q(.)}=\|f\|_{p(.),q(.)}+\|\nabla f\|_{p(.),q(.)}$.

Proposition 10. Let $p(.),q(.)\in\mathcal{P}_1([0,l])$ and $p(0),p(\infty)>1$. If $f\epsilon W^{1,p(.),q(.)}(\Omega)$, then $Mf\in W^{1,p(.),q(.)}(\Omega)$ and $|\nabla Mf(x)|\leq M|\nabla f(x)|$ for almost everywhere.

Proof. Since $L^{p(.),q(.)}(\Omega)\hookrightarrow L^1_{loc}(\Omega)$, then we can write $L^{p(.),q(.)}(\Omega)\hookrightarrow L^{p(.),q(.)}_{loc}(\Omega)\hookrightarrow L^1_{loc}(\Omega)$ and $W^{1,p(.),q(.)}(\Omega)\hookrightarrow W^{1,p(.),q(.)}_{loc}(\Omega)\hookrightarrow W^{1,1}_{loc}(\Omega)$. Since $f\in W^{1,1}_{loc}(\Omega)$, then we have $|\nabla Mf(x)|\leq M|\nabla f(x)|$ for almost everywhere in Ω by [15]. Since $f,|\nabla f|\epsilon L^{p(.),q(.)}(\Omega)$, then $Mf,\nabla Mf\epsilon L^{p(.),q(.)}(\Omega)$. Hence $Mf\in W^{1,p(.),q(.)}(\Omega)$.

Definition 11. For $f\in L^1_{loc}(\Omega)$, we denote

$$M^{\#}_{B(x,r)}f=\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)-f_{B(x,r)}|\,dy.$$

The sharp maximal function $M^{\#}f$ of f is defined by

$$M^{\#}f\,(\mathrm{x})=sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)-f_{B(x,r)}|\,dy,\ x\epsilon\Omega,$$

where $f_{B(x,r)}=\frac{1}{|B(x,r)|}\int_{B(x,r)}f(z)\,dz$.

Let 0<α<n. The fractional maximal function is defined by

$$M^{\alpha}f(x)=sup_{r>0}\frac{1}{|B(x,r)|^{1-\frac{\alpha}{n}}}\int_{B(x,r)}|f(y)|\,dy,\ x\epsilon\Omega.$$

For $f\in C_0^{\infty}(\Omega)$ or f measurable with $f\geq 0$, we define the Riesz potentials $I_{\alpha}f:\Omega\to[0,\infty]$ by

$$I_{\alpha}f(x)=\int_{\Omega}\frac{f(y)}{|x-y|^{n-\alpha}}dy.$$

Since $M^{\alpha}f(x)\leq cI_{\alpha}(f)(x)$, then the fractional maximal function M^{α} is bounded on $L^{p(.),q(.)}(\Omega)$ [14].

Proposition 12. Let $p(.),q(.)\in\mathcal{P}_1([0,l])$ and $p(0),p(\infty)>1$. Then $M^{\#}$ is bounded on $L^{p(.),q(.)}(\Omega)$, i.e. there exists a $C>0$ such that

$$\|M^{\#}f\|_{p(.),q(.)}\leq C\|f\|_{p(.),q(.)}$$

for all $f\in L^{p(.),q(.)}(\Omega)$.

Proof. If we use the inequality $|M^{\#}f|\leq 2Mf$ and the boundedness of Mf on $L^{p(.),q(.)}(\Omega)$,

$$\|M^{\#}f\|_{p(.),q(.)}\leq 2\|Mf\|_{p(.),q(.)}\leq C\|f\|_{p(.),q(.)}.$$

Proposition 13. Let $p(.),q(.)\in\mathbb{P}_1([0,\infty])$, $q(\infty)\leq p(\infty)$ and $p(0)=q(0)$. Then $C_0^{\infty}(\Omega)$ is dense in $W^{k,p(.),q(.)}(\Omega)$.

Proof. Using Proposition 7 and the definition of $W^{k,p(.),q(.)}(\Omega)$ we have the desired result.

The following theorem was proved similarly to Theorem 5.2 in [10].

Theorem 13. Let $p(.),q(.)\in\mathbb{P}_1([0,\infty])$, $q(\infty)\leq p(\infty)$ and $p(0)=q(0)$, $\frac{1}{p^*(.)}=\frac{1}{p(.)}-\frac{1}{n}$, $1<p(.)\leq p^+<n$ and $1<q(.)\leq q^+<\infty$. Also assume that for all $f\in L^{p(.),q(.)}(\Omega)$ there holds

$$\|f\|_{p(.),q(.)}\leq\|M^{\#}f\|_{p(.),q(.)} \qquad (4)$$

for $C>0$. Then $W^{1,p(.),q(.)}(\Omega)\hookrightarrow L^{\frac{np(.)}{n-p(.)},q(.)}(\Omega)$.

Proof. Let $f\epsilon W^{1,p(.),q(.)}(\Omega)$ and $\|f\|_{1,p(.),q(.)}\leq 1$. We will show $\|f\|_{p^*(.),q(.)}\leq C$. Since $C_0^{\infty}(\Omega)$ is dense in $W^{1,p(.),q(.)}(\Omega)$ and $L^{p(.),q(.)}(\Omega)$, then we can assume without loss of generality $f\in C_0^{\infty}(\Omega)$. Due to Theorem 3.15 in [14] there holds $\|I_1(|\nabla f|)\|_{p^*(.),q(.)}\leq C$. From Corollary 1.64 of [20] we deduce that for all $B(x,r)$ there holds

$$M^{\#}_{B(x,r)}f\leq Cr\frac{1}{|B(x,r)|}\int_{B(x,r)}|\nabla f(y)|\,dy$$
$$\leq C\int_{B(x,r)}\frac{|\nabla f(y)|}{|x-y|^{n-1}}dy$$
$$\leq CI_1(|\nabla f|)(x).$$

By taking the supremum over all balls $B(x,r)$ we deduce that for all x there holds

$$M^{\#}f\,(\mathrm{x})\leq CI_1(|\nabla f|)(x).$$

This and $\|I_1(|\nabla f|)\|_{p^*(.),q(.)}\leq C$ imply $\|M^{\#}f\|_{p^*(.),q(.)}\leq C$. From (4) there follows $\|f\|_{p^*(.),q(.)}\leq C$. Since $C_0^{\infty}(\Omega)$ is dense in $W^{1,p(.),q(.)}(\Omega)$ and $L^{p^*(.),q(.)}(\Omega)$, this proves $\|f\|_{p^*(.),q(.)}\leq C$ for all f with $\|f\|_{1,p(.),q(.)}\leq 1$. This proves the theorem.

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