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An Exact Variance Expression of Pisarenko's Method for Frequency Estimation of a Single Real Sinusoid: Random Phase Case

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Abstract – Several statistical analyses of the Pisarenko harmonic decomposition (PHD) method for frequency estimation of a single real sinusoid in noise have been carried out in the literature. However, all of the analyses consider the phase of the sinusoid as a constant parameter which does not change from one snapshot to another. In this paper, it is assumed that the phase is a random variable uniformly distributed in an interval of length 2π. With the use of a simple variance analysis technique, an exact expression of the PHD frequency variance is derived. An approximate PHD variance formula for sufficiently large data lengths and signal-to-noise ratios is also given. Computer simulations are included to validate the theoretical development.

Keywords – Frequency estimation; Real sinusoid; Pisarenko's method; Variance analysis.

Pisarenko Metodunun Bir Reel Sinüsün Frekans Kestirimindeki Değişintisinin Bir Kesin İfadesi: Rasgele Faz Durumu

Özetçe – Pisarenko harmonic ayrışım (PHD) metodunun gürültü içindeki bir reel sinüsün frekans kestirimindeki istatistiksel analizi birçok makalede yapılmıştır. Fakat bütün bu analizler sinüsün fazını gerçeklemeden gerçeklemeye değişmeyen sabit bir parametre biçiminde ele almıştır. Bu makalede, fazın 2π uzunluklu bir aralıkta düzgün dağılıma sahip bir rasgele değişken olduğu varsayılmıştır. Basit bir değişinti analiz tekniği kullanılarak, PHD frekans değişintisinin bir kesin ifadesi çıkarılmıştır. Yeteri kadar geniş veri uzunlukları ve yüksek işaret gürültü oranları için bir yaklaşık PHD değişinti formülü de verilmektedir. Kuramsal sonuçları teyit eden bilgisayar benzetimleri dâhil edilmiştir.

Anahtar Kelimeler – Frekans kestirimi; Reel sinüs; Pisarenko metodu; Değişinti analizi.

I. Introduction

The problem of sinusoidal frequency estimation has been frequently studied in the signal processing literature due to its wide range of applications. Among the various estimation techniques, the so-called Piseranko harmonic decomposition (PHD) method has been attracted people because of its ease of implementation. Several statistical analyses of the PHD method for frequency estimation of a single real sinusoid in noise have been carried out in the literature (see, e.g., [1], [2], and the references therein). A number of simple approximate expressions for the variance of the PHD frequency estimator have been derived (see, e.g, [1]); but these expressions are not valid for small number of data points and/or high signal-to-noise ratios (SNRs). More recently, an exact expression of the PHD frequency variance, which holds for moderate data lengths and/or SNRs, has been given in [2]. However, all of the above-mentioned analyses consider the phase of the sinusoid as a constant parameter which does not change from one snapshot to another. In this paper, it is assumed that the phase of the sinusoid is a random variable which is uniformly distributed in an interval of length 2π. By utilizing the variance analysis technique employed in [2], but taking into account that the phase is now a random variable, we derive an exact closed-form expression for the frequency variance of the PHD method. An approximate variance formula that holds for sufficiently large data lengths and SNRs is also given.

The data model and the PHD frequency estimator are described in Section II. The PHD variance development for the random phase case considered herein is given in Section III. Computer simulations are presented in Section IV to validate the theoretical development. Finally, conclusions are drawn in Section V.

II. DATA MODEL AND PHD ESTIMATOR

We consider the following data model consisting of a real single sinusoid in noise:

$$x(n) = \alpha \cos(\omega n + \phi) + e(n), \qquad n = 1, 2, \ldots, N$$

where the amplitude α and the frequency $\omega \in (0, \pi)$ of the sinusoid are unknown constants while the phase ϕ of the sinusoid is assumed to be a random variable which is uniformly distributed in the interval $(-\pi, \pi]$. The noise $e(n)$ is assumed to be a zero-mean white Gaussian process with variance σ^2 and is independent of the phase ϕ. N denotes the number of available data samples.

The PHD frequency estimator is given by [2]

$$\hat{\omega} = \arccos\left(\frac{r_2 + \sqrt{r_2^2 + 8r_1^2}}{4r_1} \right)$$

where

$$r_k = \frac{1}{N-k} \sum_{n=1}^{N-k} x(n)x(n+k), \qquad k = 1, 2$$

are the unbiased autocorrelation function estimates calculated from $x(n)$.

III. EXACT PHD FREQUENCY VARIANCE DEVELOPMENT

Let $\lambda = \cos(\hat{\omega})$ and define a quadratic function

$$f(\rho) = 2r_1\rho^2 - r_2\rho - r_1 \tag{1}$$

for which λ is one of the roots. For sufficiently large N and/or SNR, which is defined as $\mathrm{SNR} = \alpha^2/(2\sigma^2)$, λ will be close to $\cos(\omega)$ and we can use the following formula to derive the variance of λ [2] (also see [3]):

$$\mathrm{var}(\lambda) \approx \left. \frac{E\{f^2(\rho)\}}{(E\{f'(\rho)\})^2} \right|_{\rho=\cos(\omega)}. \tag{2}$$

The relationship between the variance of $\hat{\omega}$ and the variance of λ is given as [4]

$$\mathrm{var}(\hat{\omega}) \approx \frac{\mathrm{var}(\lambda)}{\sin^2(\omega)}. \tag{3}$$

Here E denotes the expectation operator and $f'(\rho)$ is the derivative of $f(\rho)$ with respect to ρ. The derivations of the expectations in (2) are given as follows. From (1) we have

$$\begin{aligned} E\{f^2(\rho)\}\big|_{\rho=\cos(\omega)} &= \cos^2(2\omega)E\{r_1^2\} - 2\cos(\omega)\cos(2\omega)E\{r_1r_2\} \\ &\quad + \cos^2(\omega)E\{r_2^2\} \end{aligned} \tag{4}$$

and

$$E\{f'(\rho)\}\big|_{\rho=\cos(\omega)} = 4\cos(\omega)E\{r_1\} - E\{r_2\}. \tag{5}$$

We see that the values of $E\{r_1\}$, $E\{r_2\}$, $E\{r_1^2\}$, $E\{r_1r_2\}$ and $E\{r_2^2\}$ are needed. These are derived as

$$E\{r_1\} = \frac{\alpha^2\cos(\omega)}{2} \tag{6}$$

$$E\{r_2\} = \frac{\alpha^2\cos(2\omega)}{2} \tag{7}$$

$$\begin{aligned} E\{r_1^2\} &= \frac{\sigma^4}{N-1} + \frac{\alpha^4\cos^2(\omega)}{4} + \frac{\alpha^4\sin^2((N-1)\omega)}{8(N-1)^2\sin^2(\omega)} + \frac{\alpha^2\sigma^2}{N-1} \\ &\quad + \frac{\alpha^2\sigma^2(N-2)\cos(2\omega)}{(N-1)^2} \end{aligned} \tag{8}$$

$$\begin{aligned} E\{r_1r_2\} &= \frac{\alpha^4\cos(\omega)\cos(2\omega)}{4} + \frac{\alpha^4\sin((N-1)\omega)\sin((N-2)\omega)}{8(N-1)(N-2)\sin^2(\omega)} \\ &\quad + \frac{\alpha^2\sigma^2\cos(\omega)}{N-1} + \frac{\alpha^2\sigma^2(N-3)\cos(3\omega)}{(N-1)(N-2)} \end{aligned} \tag{9}$$

$$\begin{aligned} E\{r_2^2\} &= \frac{\sigma^4}{N-2} + \frac{\alpha^4\cos^2(2\omega)}{4} + \frac{\alpha^4\sin^2((N-2)\omega)}{8(N-2)^2\sin^2(\omega)} + \frac{\alpha^2\sigma^2}{N-2} \\ &\quad + \frac{\alpha^2\sigma^2(N-4)\cos(4\omega)}{(N-2)^2}. \end{aligned} \tag{10}$$

Substituting (6)-(10) into (4) and (5) and using (2) and (3), after simplifications, give

$$\mathrm{var}(\hat{\omega}) \approx \frac{A(N,\omega)\cdot \mathrm{SNR}^{-2} + B(N,\omega)\cdot \mathrm{SNR}^{-1} + C(N,\omega)}{D^2(\omega)\sin^2(\omega)} \tag{11}$$

where the terms A, B, C and D are given at the top of the next page. The variance is a function of ω, N and SNR. It can be seen that A, B and C are of order N^{-1}, N^{-2} and N^{-2}, respectively, while D is not a function of N.

$$A(N,\omega)=\frac{(2N-3)+(N-1)\cos(2\omega)+(N-2)\cos(4\omega)}{8(N-1)(N-2)}$$

$$B(N,\omega)=\frac{4(N-1)(N-2)+(N-4)\cos(2\omega)-4(N-1)\cos(4\omega)-N\cos(6\omega)}{8(N-1)^2(N-2)^2}$$

$$C(N,\omega)=\frac{((N-1)\sin((N-1)\omega)+\sin((N-3)\omega)-(N-2)\sin((N+1)\omega))^2}{32(N-1)^2(N-2)^2\sin^2(\omega)}$$

$$D(\omega)=\frac{2+\cos(2\omega)}{2}$$

The expression in (11) gives the PHD frequency variance as a simple function of the SNR. For low values of SNR the first term of (11) becomes dominant and the variance is inversely proportional to the second power of the SNR. For high values of the SNR the last term in the numerator of (11) becomes dominant and the variance becomes independent of the SNR. This unusual constant behavior of the PHD variance with respect to the SNR for high SNR regimes has also been observed for the constant phase case [2].

The variance expression in [2] derived for the constant phase case depends on the phase ϕ of the sinusoid in a very complicated way; see [2, equation (15)]. In contrast, our variance expression (11) derived for the random phase case is independent of ϕ, as expected; by assuming a random phase we get rid of all the possible phase dependent terms.

For $N>>1$, the terms A, B and C can be approximated as

$$\tilde{A}(N,\omega)=\frac{\cos^2(\omega)+\cos^2(2\omega)}{4N}$$

$$\tilde{B}(N,\omega)=\frac{1}{2N^2}$$

$$\tilde{C}(N,\omega)=\frac{\cos^2(N\omega)}{8N^2}$$

respectively, and consequently a simple approximate expression of $\mathrm{var}(\hat{\omega})$ in this case is

$$\mathrm{var}(\hat{\omega})\approx\frac{\tilde{A}(N,\omega)\cdot\mathrm{SNR}^{-2}+\tilde{B}(N,\omega)\cdot\mathrm{SNR}^{-1}+\tilde{C}(N,\omega)}{D^2(\omega)\sin^2(\omega)}\ . \quad (12)$$

IV. NUMERICAL EXAMPLES

Computer simulations had been performed in order to validate our theoretical results. The amplitude of the sinusoid was set to 2 while different SNRs were obtained by properly scaling the noise samples. All simulation results were obtained by averaging 1000 independent runs.

Figures 1-3 show the frequency variances of the PHD estimator for $\omega\in[0.025\pi,0.975\pi]$ at $\mathrm{SNR}=20$ dB and for $N=8$, 32 and 128, respectively. The variance expressions of (11) and (12) were also included. We observe that the measured variances conformed the exact expression (11) even for a small value of N such as $N=8$ provided that the frequency ω is not very near to 0 or π. Also, the simple approximate expression (12), whose derivation assumed $N>>1$, was in a good agreement with the measured ones even for moderate values of N such as $N=32$.

Figures 4-6 show the PHD frequency variances versus SNR at $\omega=3\pi/8$ and for $N=8$, 32 and 128, respectively. It can be observed that (11) agreed well with the simulation results provided that $(N\cdot\mathrm{SNR})$ is not too small while (12) was a good approximation for $N\geq 32$.

V. CONCLUSIONS

We have derived an exact closed-form variance expression of the Pisarenko's method for a single real sinusoid in additive white Gaussian noise for a random phase case. An approximate simple variance formula for sufficiently large data lengths and high signal-to-noise ratios has also been developed. Computer simulations have been provided to validate the theoretical results.

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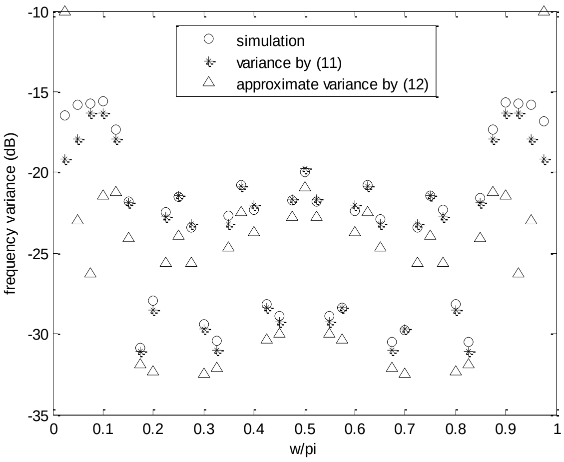

Figure 1: Frequency variances versus ω at $\text{SNR} = 20$ dB and $N = 8$.

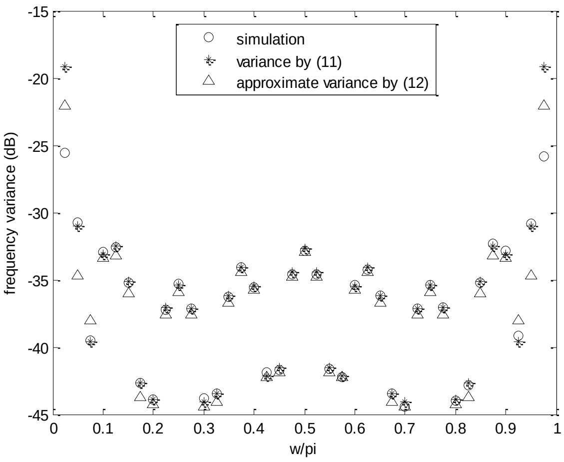

Figure 2: Frequency variances versus ω at $\text{SNR} = 20$ dB and $N = 32$.

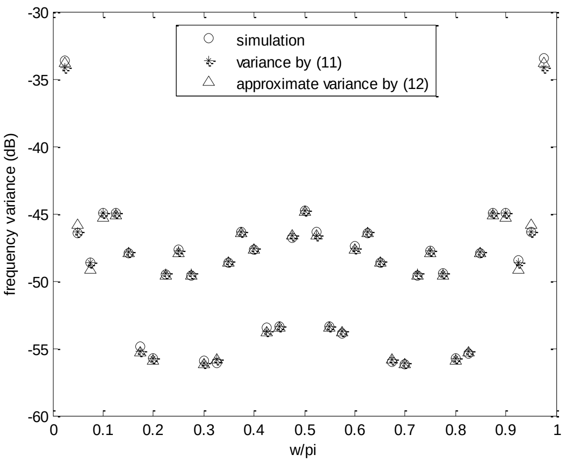

Figure: 3. Frequency variances versus ω at $\text{SNR} = 20$ dB and $N = 128$.

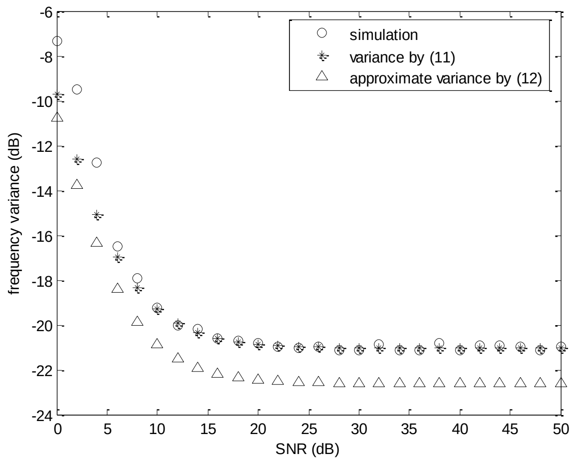

Figure 4: Frequency variances versus SNR at $\omega = 3\pi/8$ and $N = 8$.

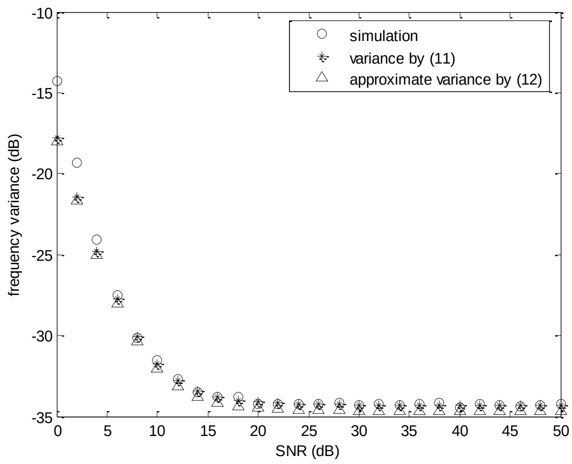

Figure 5: Frequency variances versus SNR at $\omega = 3\pi/8$ and $N = 32$.

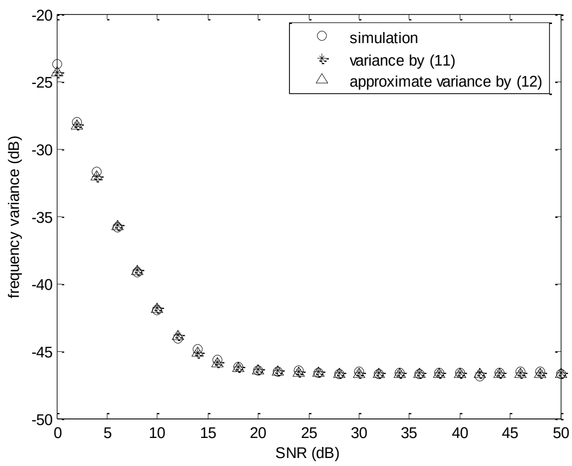

Figure 6: Frequency variances versus SNR at $\omega = 3\pi/8$ and $N = 128$.