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Simple and Accurate Bias and Variance Expressions of Some Instantaneous Frequency Estimators including the DESAs

Sewim Hazal Uz[1+] and Erdoğan Dilaveroğlu[1*]

[1]*Department of Electrical and Electronic Engineering, Bursa Uludağ University, Bursa, Turkey*
[*]*Corresponding author: dilaver@uludag.edu.tr*
[+]*Speaker: uzhazal@gmail.com*
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Abstract – Instantaneous estimation of the frequency of a real sinusoid in noise from a small number of data samples is an important problem in the signal processing area. Several frequency estimators are proposed for this problem in the literature. In this paper, some of the popular ones including the discrete energy separation algorithms (DESAs) are considered. Using a Taylor series expansion technique, very simple and yet accurate closed form expressions for the bias and the variance of the estimators are derived. Computer simulations are included to validate the theoretical results.

Keywords – Instantaneous frequency estimation; Real sinusoid; DESAs; Bias and variance analysis.

DESA ve Bazı Anlık Frekans Kestiricilerinin Basit ve Kesin Yanlılık ve Değişinti İfadeleri

Özetçe – Gürültü içindeki bir reel sinüsün frekansının az sayıda veri örneklerinden anlık kestirimi işaret işleme alanında önemli bir problemdir. Bu problem için literatürde birçok frekans kestiricisi önerilmiştir. Bu makalede, ayrık enerji ayrışım algoritmaları (DESA) ve diğer bazı popüler kestiriciler ele alınmıştır. Bir Taylor seri açılımı tekniği kullanılarak, kestiricilerin yanlılık ve değişintileri için çok basit fakat kesin kapalı biçim ifadeler çıkarılmıştır. Kuramsal sonuçları teyit eden bilgisayar benzetimleri dâhil edilmiştir.

Anahtar Kelimeler – Anlık Frekans kestirimi; Reel sinüs; DESA; Yanlılık ve değişinti analizi.

I. Introduction

A statistical performance analysis of some instantaneous frequency estimators for estimating the frequency of a real sinusoid in noise is presented for high signal-to-noise ratio regimes. The frequency estimators derived from the discrete energy separation algorithms (DESA-1a, DESA-1, DESA-2) introduced in [1] and the four additional estimators proposed in [2], which are derived from the modified covariance and Prony's methods, are considered. Using a Taylor series expansion technique, very simple and yet accurate closed form expressions for the biases and the variances of the estimators are derived. A performance comparison of these estimators is also given.

The data model and the instantaneous frequency estimators considered herein are described in Section II. The bias and variance development is given in Section III. Computer simulations are presented in Section IV to validate the theoretical results. Finally, conclusions are drawn in Section V.

II. Data Model and Instantaneous Frequency Estimators

We consider the following data model consisting of a real sinusoid in noise:

$$x_n = \alpha\cos(\omega n + \phi) + e_n\,, \qquad n = 0,1,\ldots,N-1$$

where the amplitude α and the frequency $\omega \in (0,\pi)$ of the sinusoid are unknown constants while the phase ϕ of the sinusoid is assumed to be a random variable which is uniformly distributed in the interval $(-\pi,\pi]$. The noise $e(n)$ is assumed to be a zero-mean white Gaussian process with variance σ^2 and is independent of the phase ϕ. N denotes the number of data samples utilized.

The instantaneous frequency estimators considered herein require four or five data samples. The formulas for calculating the estimates from x_0, x_1, x_2, x_3 (and x_4 for the estimators that require five data samples) are shown in Table I given at the bottom of the page.

III. BIAS AND VARIANCE DEVELOPMENT

For each estimator in Table I, except the DESA-2, define $\rho = \cos(\hat{\omega})$; for DESA-2 define $\rho = \cos(2\hat{\omega})$. Next identify ρ as

$$\rho = \rho(A, B) = \frac{A}{B}.$$

For sufficiently large signal-to-noise ratio (SNR), defined as $\mathrm{SNR} = \alpha^2/(2\sigma^2)$, ρ will be close to $\cos\omega$ ($\cos 2\omega$ for DESA-2) and we can use the following formulas, which are based on a Taylor series expansion of $\rho(A,B)$, to derive the expected value and the variance of ρ [3] (also see [4]):

$$\bar{\rho} \approx \rho(A,B)\Big|_{A=\bar{A},B=\bar{B}} \tag{1}$$

$$\begin{aligned} \mathrm{var}(\rho) \approx\ & \mathrm{var}(A) \cdot \left(\frac{\partial \rho(A,B)}{\partial A}\Big|_{A=\bar{A},B=\bar{B}} \right)^2 \\ & + \mathrm{var}(B) \cdot \left(\frac{\partial \rho(A,B)}{\partial B}\Big|_{A=\bar{A},B=\bar{B}} \right)^2 \\ & + 2\,\mathrm{cov}(A,B) \cdot \frac{\partial \rho(A,B)}{\partial A} \frac{\partial \rho(A,B)}{\partial B}\Big|_{A=\bar{A},B=\bar{B}} \end{aligned} \tag{2}$$

where an over bar denotes the expectation of the quantity beneath it. The bias of each estimator $\hat{\omega}$, denoted by $\mathrm{bias}(\hat{\omega})$, is defined as $\mathrm{bias}(\hat{\omega}) = \bar{\hat{\omega}} - \omega$. The relationship between $\mathrm{bias}(\hat{\omega})$ and $\bar{\rho}$ and that between $\mathrm{var}(\hat{\omega})$ and $\mathrm{var}(\rho)$ are given as [3]

$$\mathrm{bias}(\hat{\omega}) \approx \arccos(\bar{\rho}) - \omega$$

$$\mathrm{var}(\hat{\omega}) \approx \frac{\mathrm{var}(\rho)}{1-\bar{\rho}^2}.$$

(For DESA-2 $\mathrm{bias}(\hat{\omega}) \approx \frac{\arccos(\bar{\rho})}{2} - \omega$ and $\mathrm{var}(\hat{\omega}) \approx \frac{\mathrm{var}(\rho)}{4(1-\bar{\rho}^2)}$). We see from (1) and (2) that the values of $\bar{A}$, $\bar{B}$, $\mathrm{var}(A)$, $\mathrm{var}(B)$ and $\mathrm{cov}(A,B)$ are needed for each estimator. The derivations of these terms are straightforward (albeit tedious) and are not given here due to space limitations. The final results, which are summarized in Table II shown at the top of the next page, were obtained by substituting the derived expressions of the aforementioned terms into (1) and (2), and retaining only the dominant term of (2) for large SNR.

The mean and variance expressions of ρ (and thus those of $\hat{\omega}$) are functions of the frequency ω and the SNR. It can be seen that, for large values of SNR, ρ ($\hat{\omega}$) is approximately unbiased and $\mathrm{var}(\rho)$ ($\mathrm{var}(\hat{\omega})$), which becomes the main part of the mean squared estimation error, is inversely proportional to the first power of SNR for each of the estimators.

TABLE I
INSTANTANEOUS FREQUENCY ESTIMATORS

Estimator	$\hat{\omega}$
DESA-1a (4 pt)	$\arccos\left(\frac{(x_2^2 - x_1x_3) - (x_1^2 - x_0x_2) + (x_1x_2 - x_0x_3)}{2(x_2^2 - x_1x_3)}\right)$
Modified Covariance (4 pt)	$\arccos\left(\frac{x_0x_1 + 2x_1x_2 + x_2x_3}{2(x_1^2 + x_2^2)}\right)$
Prony (4 pt)	$\arccos\left(\frac{x_1x_2 - x_0x_3}{(x_1^2 - x_0x_2) + (x_2^2 - x_1x_3)}\right)$
DESA-1 (5 pt)	$\arccos\left(\frac{2(x_2^2 - x_1x_3) - (x_1^2 - x_0x_2) - (x_3^2 - x_2x_4) + x_1x_2 - x_0x_3 + x_2x_3 - x_1x_4}{4(x_2^2 - x_1x_3)}\right)$
DESA-2 (5 pt)	$\frac{1}{2}\arccos\left(\frac{(x_2^2 - x_0x_4) - (x_1^2 - x_0x_2) - (x_3^2 - x_2x_4)}{2(x_2^2 - x_1x_3)}\right)$
Modified Covariance (5 pt)	$\arccos\left(\frac{x_0x_1 + 2x_1x_2 + 2x_2x_3 + x_3x_4}{2(x_1^2 + x_2^2 + x_3^2)}\right)$
Modified Prony (5 pt)	$\arccos\left(\frac{(x_1x_2 - x_0x_3) + (x_2x_3 - x_1x_4)}{4(x_2^2 - x_1x_3)}\right)$

TABLE II
APPROXİMATE EXPRESSIONS OF $\bar{\rho}$ AND $\mathrm{var}(\rho)$

Estimator	$\bar{\rho}$	$\mathrm{var}(\rho)$
DESA-1a (4 pt)	$\frac{2\mathrm{SNR}\sin^2\omega\cos\omega}{1+2\mathrm{SNR}\sin^2\omega}$	$\frac{3+2\cos2\omega}{4\mathrm{SNR}\sin^2\omega\cos^2(\omega/2)}$
Modified Covariance (4 pt)	$\frac{\mathrm{SNR}\cos\omega}{\mathrm{SNR}+1}$	$\frac{1}{4\mathrm{SNR}}$
Prony (4 pt)	$\frac{2\mathrm{SNR}\sin^2\omega\cos\omega}{1+2\mathrm{SNR}\sin^2\omega}$	$\frac{3+2\cos2\omega}{4\mathrm{SNR}\sin^2\omega}$
DESA-1 (5 pt)	$\frac{2\mathrm{SNR}\sin^2\omega\cos\omega}{1+2\mathrm{SNR}\sin^2\omega}$	$\frac{12-14\cos\omega+8\cos2\omega-\cos3\omega}{32\mathrm{SNR}\sin^2\omega\cos^2(\omega/2)}$
DESA-2 (5 pt)	$\frac{4\mathrm{SNR}\sin^2\omega\cos2\omega-1}{2(1+2\mathrm{SNR}\sin^2\omega)}$	$\frac{4+\cos2\omega}{2\mathrm{SNR}\sin^2\omega}$
Modified Covariance (5 pt)	$\frac{\mathrm{SNR}\cos\omega}{\mathrm{SNR}+1}$	$\frac{1}{9\mathrm{SNR}}$
Modified Prony (5 pt)	$\frac{2\mathrm{SNR}\sin^2\omega\cos\omega}{1+2\mathrm{SNR}\sin^2\omega}$	$\frac{5(4+3\cos2\omega)}{32\mathrm{SNR}\sin^4\omega}$

IV. NUMERICAL EXAMPLES

Computer simulations had been performed in order to validate our theoretical results. The amplitude of the sinusoid was set to 2 while different SNRs were obtained by properly scaling the noise samples. All simulation results were obtained by averaging 1000 independent runs.

Figures 1-2 show the frequency variances of the four and five-point estimators, respectively, for $\omega \in [0.1\pi, 0.9\pi]$ at $\mathrm{SNR}=30$ dB. The theoretical variance expressions from Table II were also included. We observe that the measured variances conformed the theoretical expressions, provided that the frequency ω is not near to 0 or π. It can be seen from Figure 1 that, among the four-point estimators, the relation $\mathrm{var}(\hat{\omega}_{\mathrm{DESA-1a}}) > \mathrm{var}(\hat{\omega}_{\mathrm{Prony}}) > \mathrm{var}(\hat{\omega}_{\mathrm{Mod.\,Cov.}})$ holds for the whole range of ω. We see from Figure 2 for the five-point estimators that again the variance of the modified covariance method is less than that of DESA-1 or the modified Prony. Also, while $\mathrm{var}(\hat{\omega}_{\mathrm{Mod.\ Prony}}) > \mathrm{var}(\hat{\omega}_{\mathrm{DESA-1}})$ for $\omega \le 0.46\pi$, this relation reverses for $\omega > 0.46\pi$.

Figures 3-4 show the frequency variances of the four and five-point estimators, respectively, versus SNR at $\omega = 3\pi/8$. It can be observed that the theoretical expressions agreed well with the simulation results for $\mathrm{SNR} \ge 15$ dB.

Note that the five-point estimator DESA-2 was not included in the above experiments because it does not work for $\omega \in (\pi/2, \pi)$. The measured and theoretical frequency variances of DESA-2 are shown in Figure 5 for $\omega \in [0.05\pi, 0.45\pi]$ at $\mathrm{SNR}=30$ dB and Figure 6 shows the variances versus SNR at $\omega = 3\pi/16$. Figures 2 and 5 may be used for a variance comparison of the DESA-2 and the other five-point estimators. It can be seen that the variance of DESA-2 falls in between those of the modified Prony and DESA-1 for $\omega < 0.35\pi$ but it is greater than those of all the other estimators for $\omega \ge 0.35\pi$.

V. CONCLUSIONS

A statistical analysis of seven instantaneous frequency estimators has been presented for estimating the frequency of a real sinusoid in additive white Gaussian noise. Very simple and yet accurate closed form expressions for the bias and variance of the estimators have been derived for sufficiently high signal-to-noise ratios. A performance comparison of the estimators has also been given. Computer simulations have been provided to validate the theoretical results.

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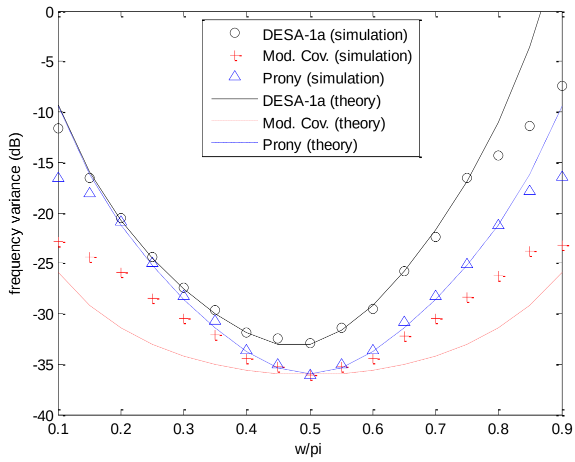

Figure 1: Frequency variances of four-point estimators versus ω at $\text{SNR} = 30$ dB.

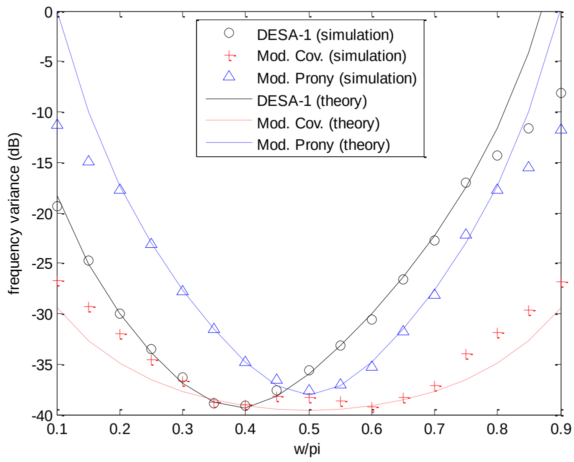

Figure: 2. Frequency variances of five-point estimators versus ω at $\text{SNR} = 30$ dB.

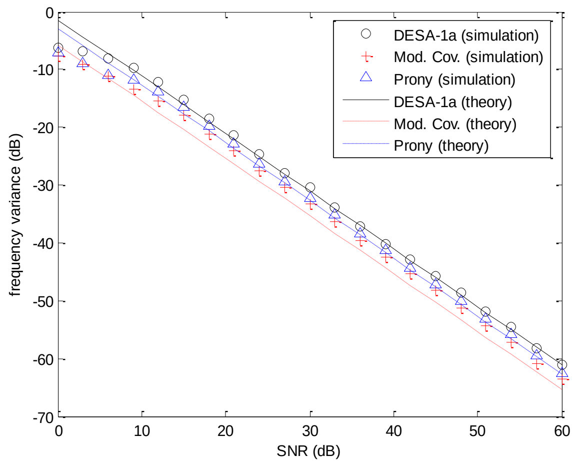

Figure 3: Frequency variances of four-point estimators versus SNR at $\omega = 3\pi/8$.

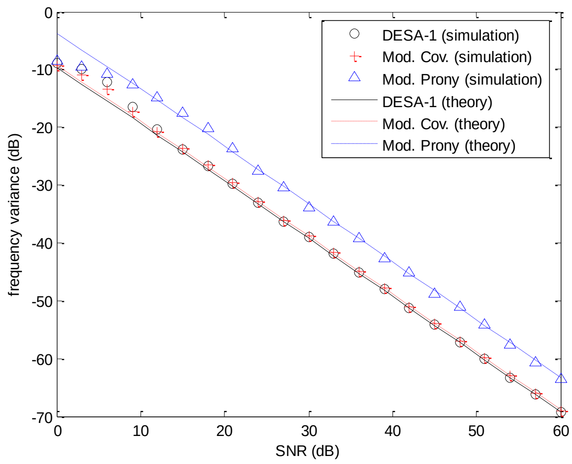

Figure 4: Frequency variances of five-point estimators versus SNR at $\omega = 3\pi/8$.

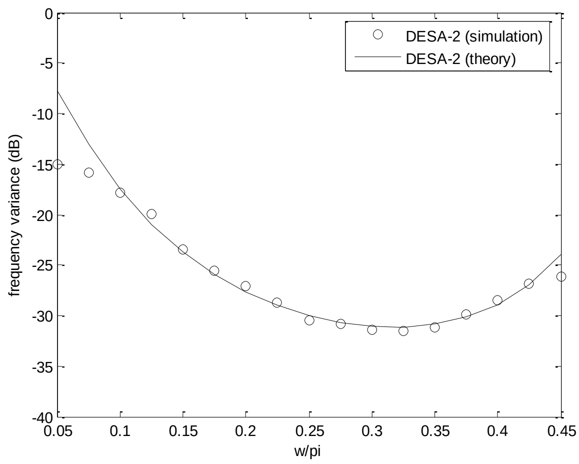

Figure 5: Frequency variances of DESA-2 versus ω at $\text{SNR} = 30$ dB.

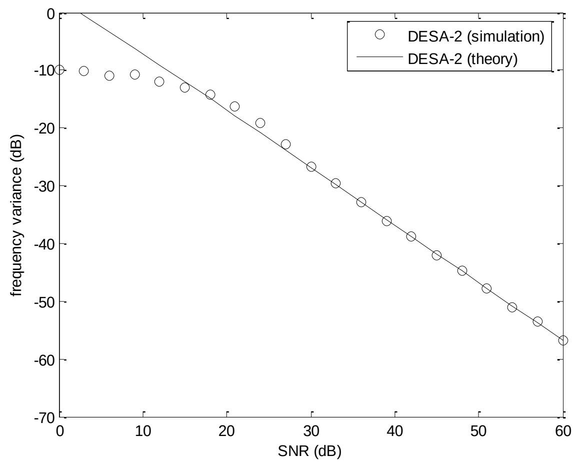

Figure 6: Frequency variances of DESA-2 versus SNR at $\omega = 3\pi/16$.